1. Let a, b, c be numbers, with $a \neq 0$. Let α be a number. Let $f(x)$ be the quadratic polynomial given by $ax^2 + bx + c$.

   (a) Suppose α is a root of $f(x)$. Let $\beta = -\dfrac{b}{a} - \alpha$. Prove the statements below:

   i. $f(x) = a(x - \alpha)(x - \beta)$ as polynomials.

   ii. β is a root of $f(x)$.

   iii. $\alpha\beta = \dfrac{c}{a}$.

   Remark. Up to this point, we have not addressed the question whether $f(x)$ has any root (and where it is) in the first place.

   (b) Define $\Delta_f = b^2 - 4ac$.

   i. Verify that $f(x) = a\left[\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{\Delta_f}{4a^2}\right]$ as polynomials.

   ii. Suppose a, b, c are real numbers.

   A. Suppose $\Delta_f \geq 0$. Define $\alpha_\pm = \dfrac{-b \pm \sqrt{\Delta_f}}{2a}$ respectively.
   Verify that $f(x) = a(x - \alpha_+)(x - \alpha_-)$ as polynomials.

   B. Now suppose $\Delta_f < 0$ instead. Define $\zeta = \dfrac{-b + i\sqrt{-\Delta_f}}{2a}$.
   Verify that that $f(x) = a(x - \zeta)(x - \bar{\zeta})$ as polynomials.

   Remark. We have now confirmed that the quadratic polynomial with real coefficients $f(x)$ has a pair of roots and how it factorizes into linear factors.

   (c) Now we no longer suppose '*a, b, c are real numbers*'.

   i. Suppose $\Delta_f \neq 0$, and σ is a square root of $\dfrac{\Delta_f}{4a^2}$ in the complex numbers. Define $\alpha_\pm = -\dfrac{b}{2a} \pm \sigma$ respectively.
   Verify that $f(x) = a(x - \alpha_+)(x - \alpha_-)$ as polynomials.

   ii. Now suppose $\Delta_f = 0$ instead. Verify that $f(x) = a\left(x + \dfrac{b}{2a}\right)^2$.

   Remark. We have now confirmed that quadratic polynomial with complex coefficients $f(x)$ has a pair of roots and how it factorizes into linear factors.

2. Let a, b, c, r be numbers, with $a \neq 0$ and $c \neq 0$ and $r \neq 0$. Let $f(x)$ be the quadratic polynomial given by $f(x) = ax^2 + bx + c$. Suppose α, β are the roots of $f(x)$. Further suppose $\alpha = r\beta$.

   Prove that $rb^2 = (Pr + Q)^2 ac$. Here P, Q are some integers whose values you have to determined explicitly.

3. Solve for all real solutions of each of the inequalities/systems below: [1]

   (a) $x^2 - 3x < 10$.

   (b) $\begin{cases} (x+1)(x-6) & \geq & 8 \\ 3x - 1 & \geq & 5 \end{cases}$

   (c) $(x + 1)^2 > 16$ or $2x + 5 > 7$.

   (d) $(x - 1)(x - 2)(x - 3) \geq 0$.

[1] In various situations, you may need apply some special rules about the words '*and*', '*or*', known as the *Distributive Laws for 'and', 'or'*, (with or without your being aware of them). They may be in-formally stated as below:

A. *The pair of statements below are the same in the sense that one holds exactly when the other holds:*

- *(blah-blah-blah or bleh-bleh-bleh) and bloh-bloh-bloh.*
- *(blah-blah-blah and bloh-bloh-bloh) or (bleh-bleh-bleh and bloh-bloh-bloh).*

B. *The pair of statements below are the same in the sense that one holds exactly when the other holds:*

- *(blah-blah-blah and bleh-bleh-bleh) or bloh-bloh-bloh.*
- *(blah-blah-blah or bloh-bloh-bloh) and (bleh-bleh-bleh or bloh-bloh-bloh).*

More will be said of them in the discussion on *logic*.

(e) $\dfrac{2}{3-x} \leq 1$.

(f) $2x - \dfrac{3}{x} \geq 1$.

(g) $\dfrac{x^2-1}{x^2-4} \leq -2$.

(h) $|x^2 - 5x| < 6$.

(i) $\left|\dfrac{3x+11}{x+2}\right| < 2$.

(j)$^\diamondsuit$ $\big|\ |x| - 4\ \big| > 3$.

(k) $|x^2 - 3| \leq 2|x|$.

(l)$^\diamondsuit$ $|2x+1| < 3x - 2$.

Remark. Now suppose you are not required to give any step of algebraic manipulation. Can you modify the 'graphical method' for solving equations in *school mathematics* to determine the answer for each part as quickly as possible?

4. Let p be a real number. Let $f(x)$ be the quadratic polynomial given by $f(x) = x^2 + (p+1)x + (p-1)$. Suppose α, β are the roots of $f(x)$.

   (a) Prove that α, β are real and distinct.

   (b) Express $(\alpha - 2)(\beta - 2)$ in terms of p.

   (c) Suppose $\beta < 2 < \alpha$.

      i. Prove that $p < -\dfrac{5}{3}$.

      ii. Further suppose $(\alpha - \beta)^2 < 20$. Prove that $-3 < p < -\dfrac{5}{3}$.

5. (a) Let $n \in \mathbb{N}$. Prove that $\sqrt{n+2} - \sqrt{n+1} < \dfrac{1}{2\sqrt{n+1}} < \sqrt{n+1} - \sqrt{n}$.

   Remark. There is no need for mathematical induction.

   (b) Hence prove that $193 < \displaystyle\sum_{k=10}^{10000} \frac{1}{\sqrt{k}} < 194$.

6. (a) Consider the statement (A):

   (A) *Let $x, y \in \mathbb{R}$. Suppose $x + y > 1$ and $x > y$. Then $x^2 - y^2 > x - y$.*

   Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (A). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   > Let $x, y \in \mathbb{R}$. ________(I)________ .
   > ___(II)___ $x + y > 1$, we have $x + y - 1 > 0$.
   > Since ____(III)____ , we have $x - y > 0$.
   > Then $(x^2 - y^2) - (x - y) =$ ________(IV)________ > 0.
   > Therefore ______(V)______ .

   (b) Consider the statement (B):

   (B) *Let $x, y \in \mathbb{R}$. Suppose $x > 0$ and $y > 0$. Then $x^3 + y^3 \geq xy(x+y)$.*

   Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (B). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   > ____________(I)____________ .
   > Then $x + y > 0$. Also, $(x - y)^2 \geq 0$.
   > We have $(x^3 + y^3) - xy(x + y) =$ ______________(II)______________ .
   > Therefore $x^3 + y^3 \geq xy(x + y)$.

7. Consider the statement (C):

(C) *Let* $x, y, z \in \mathbb{R}$. *Suppose* $y > x > 0$ *and* $z > -y$. *Then* $\dfrac{x+z}{y+z} > \dfrac{x}{y}$ *iff* $z > 0$.

Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (C). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

> Let $x, y, z \in \mathbb{R}$. ______(I)______ .
>
> Since __(II)__ , we have $y + z > 0$. Then, since $y > 0$ also, we have $y(y + z)$ __(III)__ .
>
> - __(IV)__ $z > 0$.
>
>   Then we have ______(V)______ . Therefore $(x+z)y = xy+zy > xy+zx = x(y+z)$.
>
>   Hence ______(VI)______ .
>
>   Therefore $\dfrac{x+z}{y+z} > \dfrac{x}{y}$.
>
> - ______(VII)______ .
>
>   Then $xy + zy = (x + z)y =$ ______(VIII)______ $= x(y + z) = xy + zx$.
>
>   ______________(IX)______________ .
>
>   Hence $z > 0$.
>
> It follows that ______________(X)______________ .

8.◇ Prove the statement below:

- *Let* $m, n \in \mathbb{N}\backslash\{0\}$. *Let* x *be a positive real number. Suppose* $m > n$. *Then* $x^m + \dfrac{1}{x^m} \geq x^n + \dfrac{1}{x^n}$. *Moreover, equality holds iff* $x = 1$.

9. (a) Prove the statement (♯) below:

   (♯) *Let* $u, v, x, y \in \mathbb{R}$. $(ux + vy)^2 \leq (u^2 + v^2)(x^2 + y^2)$.

   Remark. This is a 'baby version' of the Cauchy-Schwarz Inequality.

   (b) Hence, or otherwise, prove the statement (♭) below:

   (♭) *Let* s, t *be positive real numbers.* $(s + t)\left(\dfrac{1}{s} + \dfrac{1}{t}\right) \geq 4$.

10. (a)◇ By considering the non-negativity of squares, or otherwise, prove the statement (♯) below:

   (♯) *Let* a, b, c, d *be positive real numbers.* $\dfrac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$.

   (b)♣ Hence, or otherwise, prove the statement (♭) below:

   (♭) *Let* r, s, t *be positive real numbers.* $\dfrac{r+s+t}{3} \geq \sqrt[3]{rst}$.

   Remark. These are 'baby versions' of the Arithmetico-Geometrical Inequality.

11.◇ Let c, ε be positive real numbers. Define $\delta = \sqrt{c^2 + \varepsilon} - c$.

   (a) Prove that $\delta > 0$.

   (b) Let x be a real number. Suppose $|x - c| < \delta$.

      i. Prove that $|x + c| \leq \sqrt{c^2 + \varepsilon} + c$.

      ii. Hence, or otherwise, deduce that $|x^2 - c^2| < \varepsilon$.

   Remark. This is what we have verified overall: *For any* $c > 0$, *for any* $\varepsilon > 0$, *there exists some* $\delta > 0$, *(namely,* $\delta = \sqrt{c^2 + \varepsilon} - c$*) such that for any* $x \in \mathbb{R}$, *if* $|x - c| < \delta$ *then* $|x^2 - c^2| < \varepsilon$. Hence we have argued for the continuity of the function t^2 at every positive value of t.

12. Let $n \in \mathbb{N} \backslash \{0\}$.

(a)$^\diamondsuit$ Let $a \in \mathbb{R}$. Suppose $a > 1$. Prove that $(n+1)(a-1) < a^{n+1} - 1 < (n+1)a^n(a-1)$.

(b)$^\clubsuit$ Hence prove the statement below:

- *Let $b \in \mathbb{R}$. Suppose $b > 1$. Then $b^{n+1} - (b-1)^{n+1} < (n+1)b^n < (b+1)^{n+1} - b^{n+1}$.*

(c)$^\diamondsuit$ Hence prove the statement below:

- *Suppose $m \in \mathbb{N} \backslash \{0, 1\}$. Then* $\dfrac{m^{n+1}}{n+1} < \displaystyle\sum_{k=1}^{m} k^n < \dfrac{(m+1)^{n+1} - 1}{n+1}$.