MATH1050 Exercise 1 Supplement

1. Solve for all real solutions of each of the equations below:

   (a) $\dfrac{4x-7}{3x+5} = \dfrac{5}{3}$.

   (b) $\dfrac{x}{x-2} - \dfrac{x+1}{x-1} = \dfrac{x-8}{x-6} - \dfrac{x-9}{x-7}$.

   (c) $\dfrac{x^2-1}{x^2+1} = \dfrac{1}{2}$.

   (d) $\dfrac{1}{x^3-x^2-x+1} + \dfrac{1}{x^3-3x^2-x+3} = \dfrac{2}{x^3-x^2-2x}$.

2. Solve for all real solutions of each of the equations below:

   (a) $\sqrt{2x+9} = x-3$.

   (b) $\sqrt{2x-3} = \sqrt{1-2x}$.

   (c) $\sqrt{x} - \dfrac{6}{\sqrt{x}} = 1$.

   (d) $\sqrt{x} - \sqrt{x-2} = 1$.

   (e) $\sqrt{2x+3} + \sqrt{x+1} = \sqrt{8x+1}$.

   (f) $\sqrt{5x+1} + \sqrt{x+1} = \sqrt{10x+6}$.

   (g) $\sqrt{x^2+5x+2} = 1 + \sqrt{x^2+5}$.

   (h) $\dfrac{\sqrt{x}+9}{\sqrt{x}-6} = \dfrac{\sqrt{x}-5}{\sqrt{x}-13}$.

   (i) $\dfrac{1}{\sqrt{x^2-1}-x} + \dfrac{1}{\sqrt{x^2-1}+x} = -8$.

3. Solve for all real solutions of each of the equations below:

   (a) $3^{2x+1} - 25 \cdot 3^x - 18 = 0$.

   (b) $5^{x+1} + 4 \cdot 5^{1-x} = 25$.

   (c) $2^{(x^2-1)} \cdot 3^{2x-3} = 24$.

   (d) $\ln(x) + \ln(2x-1) = 0$.

   (e) $\log_{10}(x^2+1) - \log_{10}(x-2) = 1$.

   (f) $\log_2(x) - \log_x(8) = 2$.

   (g) $\log_{10}(x^2+9) - 2\log_{10}(x) = 1$.

   (h) $\log_2(x+1) + \log_2(x+4) = 1 + 2\log_2(3)$.

   (i) $\log_3(\log_2(x)) + 2\log_9(\log_7(8)) = 2$.

   (j) $(\ln(x))^2 = \ln(x^2)$.

   (k) $2\ln(x^{\ln(x)}) + 5\ln(x) = 3$.

4. Solve for all real solutions of each of the equations below:

   (a) $|3x-5| = 31$.

   (b) $3|x-2| = 10$.

   (c) $|2-1/x| = 3$.

   (d) $|x^2-5x| = 6$.

   (e) $|x^2+x-13| = 7$.

   (f) $|x^2-5x+2| = 2$.

   (g) $2x = |x-2|$.

   (h) $|x-1| = |x^2-4x+3|$.

   (i) $|x-3| = |x^2-4x+3|$.

   (j) $|x-1| = |x|-1$.

   (k) $|x^2-x-8| = |4x-2|$

   (l) $|x^2-4| = x-2$.

   (m) $(x-3)^2 - |x-3| - 12 = 0$.

   (n) $(x-5)^2 - 2|x-5| - 8 = 0$.

   (o)$^\diamondsuit$ $(x-1)|x| = x|x-1|$.

5. Consider each of the equations below. Determine whether it has any real solution at all. Where it does, determine all its real solutions. Justify your answer.

   (a) $x = x$.

   (b) $0 \cdot x = 0$.

   (c) $\dfrac{x^2-2x+1}{x-1} = 0$.

   (d) $\dfrac{x}{x-1} = \dfrac{1}{x-1}$.

   (e) $\dfrac{x^2-1}{x-1} = 0$.

   (f) $\dfrac{x}{x} = 1$.

   (g) $\dfrac{1}{x-1} = \dfrac{1}{x-1}$.

   (h) $\dfrac{1}{x-1} = \dfrac{x+1}{x^2-1}$.

6.$^\diamondsuit$ Solve for all real solutions of each of the systems of equations below:

   (a) $\begin{cases} 3x + 2y = 5 \\ x^2 - 4xy + 3 = 0 \end{cases}$

   (b) $\begin{cases} 3x^2 - xy - y^2 = 3 \\ x + y = 9 \end{cases}$

   (c) $\begin{cases} 2x^2 - y^2 = 2y \\ 6x^2 + xy - y^2 = 8y \end{cases}$

   (d) $\begin{cases} x^2 - xy - y^2 = y \\ x^2 - 4y^2 = 0 \end{cases}$

   (e) $\begin{cases} 1/x^2 + 1/y^2 = 34 \\ 15xy = 1 \end{cases}$

   (f) $\begin{cases} x^2 + y^2 = 5 \\ 1/x^2 + 1/y^2 = 5/4 \end{cases}$

   (g) $\begin{cases} x/y + y/x = 17/4 \\ x^2 - 4xy + y^2 = 1 \end{cases}$

   (h) $\begin{cases} x - y = 3 \\ \log_{10}(x) + \log_{10}(y) = 1 \end{cases}$

   Remark. At some stage of the calculation, brute force is necessary. However, try to observe before starting any calculation how you may simplify a system before resorting to brute force.

7. Let c be a real number. Consider the equation

$$cx = c + 1 \quad \text{——} \quad (\star_c)$$

   with unknown x.

   (a) Suppose $c \neq 0$. Write down all real solutions of $(\star_c)$.

(b)♣ Suppose $c = 0$. Does $(\star_c)$ have any real solution? Justify your answer.

8. Let c be a real number. Consider the equation

$$cx = c(c+1) \quad — \quad (\star_c)$$

with unknown x.

(a) Suppose $c \neq 0$. Write down all real solutions of $(\star_c)$.

(b)♣ Suppose $c = 0$. Does $(\star_c)$ have any real solution? Justify your answer.

9. Let a, b be real numbers. Consider the equation

$$(a^2 - 4a + 3)x = b - 2 \quad — \quad (\star_{a,b})$$

with unknown x.

(a) Suppose $a^2 - 4a + 3 \neq 0$. Write down all real solutions of $(\star_{a,b})$.

(b) Suppose $a^2 - 4a + 3 = 0$.

i. Suppose $(\star_{a,b})$ has a real solution. What are the respective values of a, b?

ii.♣ Determine all the solutions of $(\star_{a,b})$ where it has any solution at all.

10. Let c be a real number. Consider the system of equations

$$(\star_c) \left\{ \begin{array}{rcrcr} x & + & 2y & = & 3 \\ 2x & + & 3y & = & 4 \\ 3x & + & cy & = & 5 \end{array} \right.$$

with unknowns x, y.

(a)◇ Suppose $(\star_c)$ has a real solution. Find all possible value(s) of c.

(b) For each such values of c, solve $(\star_c)$.

11. We introduce the definition below:

- *Let $n \in \mathbb{N}\backslash\{0,1\}$, and $a, b \in \mathbb{Z}$. a is said to be* **congruent to** b **modulo** n *if $a - b$ is divisible by n. We write $a \equiv b (\mathsf{mod}\ n)$.*

Let $n \in \mathbb{N}\backslash\{0,1\}$. Prove the following statements:

(a) *Let $x \in \mathbb{Z}$. $x \equiv x(\mathsf{mod}\ n)$.*

(b) *Let $x, y \in \mathbb{Z}$. Suppose $x \equiv y(\mathsf{mod}\ n)$. Then $y \equiv x(\mathsf{mod}\ n)$.*

(c) *Let $x, y, z \in \mathbb{Z}$. Suppose $x \equiv y(\mathsf{mod}\ n)$ and $y \equiv z(\mathsf{mod}\ n)$. Then $x \equiv z(\mathsf{mod}\ n)$.*

12. *In this question you are assumed to be familiar with one-variable calculus.*

Take for granted the validity of the statement (RT), known as **Rolle's Theorem**:

(RT) *Let $a, b \in \mathbb{R}$, with $a < b$, and h be a function defined on $[a, b]$.*
Suppose h satisfies all the conditions below:

(C) *h is continuous on $[a, b]$.* (D) *h is differentiable on (a, b).* (E) *$h(a) = h(b)$.*

Then there exists some $\zeta \in (a, b)$ such that $h'(\zeta) = 0$.

(a) Apply Rolle's Theorem to deduce the statement (MVT), known as **Mean-Value Theorem**:

(MVT) *Let $a, b \in \mathbb{R}$, with $a < b$, and f be a function defined on $[a, b]$.*
Suppose f satisfies all the conditions below:

(C) *f is continuous on $[a, b]$.* (D) *f is differentiable on (a, b).*

Then there exists some $\zeta \in (a, b)$ such that $f(b) - f(a) = (b - a)f'(\zeta)$.

(b) Apply the Mean-Value Theorem or Rolle's Theorem to deduce the statement (CT) below, known as **'Constancy Theorem'**:

(CT) *Let f be a real-valued function defined on some open interval I in $\mathbb{R}$.*
Suppose f satisfies all the conditions:

(C) f *is continuous on* I. (D) f *is differentiable on* I. (Z) $f'(x) = 0$ *for any* $x \in I$.

Then f *is constant on* I.

Remark. That a real-valued function of one real variable is constant on an interval if its first derivative throughout that interval is constant zero, has nothing to do with integration.

13. Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{C}$. Consider the statements (A), (B), (C), (D), (E) below:

(A) $\{a_n\}_{n=0}^{\infty}$ *is an arithmetic progression.*

(B) *There exists some* $d \in \mathbb{C}$ *such that for any* $n \in \mathbb{N}$, $a_n = a_0 + nd$.

(C) *For any* $k \in \mathbb{N}$, $a_{k+2} - a_{k+1} = a_{k+1} - a_k$.

(D) *For any* $k \in \mathbb{N}$, $a_{k+1} = \dfrac{a_k + a_{k+2}}{2}$.

(E) *For any* $k \in \mathbb{N}$, *the numbers* a_k, a_{k+1}, a_{k+2} *form an arithmetic progression.*

Prove the statements below:

(a) If (A) holds then (B) holds.
(b) If (B) holds then (A) holds.
(c) If (B) holds then (C) holds.
(d) If (C) holds then (D) holds.
(e) If (D) holds then (A) holds.
(f)$^\diamondsuit$ If (D) holds then (E) holds.
(g)$^\clubsuit$ If (E) holds then (C) holds.

Remark. This is how we may show that the statements (A), (B), (C), (D), (E) are logically equivalent in the sense that if one of them holds for the infinite sequence $\{a_n\}_{n=0}^{\infty}$, all other statements hold as well. Formulate an analogous result for geometric progressions and prove it as well.

14. Let a, b, c be non-zero numbers. Suppose $a+b, b+c, c+a$ are all non-zero. Suppose $\dfrac{1}{a^2}, \dfrac{1}{b^2}, \dfrac{1}{c^2}$ are in arithmetic progression. Show that $\dfrac{bc}{b+c}, \dfrac{ca}{c+a}, \dfrac{ab}{a+b}$ are in arithmetic progression.

15. Let $s, t, u, v \in \mathbb{C}\backslash\{0\}$. Suppose $\dfrac{1}{s}, \dfrac{1}{t}, \dfrac{1}{u}, \dfrac{1}{v}$ form an arithmetic progression.

(a) Prove that $t = \dfrac{2su}{s+u}$.

(b) Express $\dfrac{t}{v}$ in terms of s, u.

16. Let $u, v, w \in \mathbb{C}\backslash\{0\}$. Suppose they are pairwise distinct. Suppose $\dfrac{1}{u}, \dfrac{1}{v}, \dfrac{1}{w}$ form an arithmetic progression. Also suppose u, w, v form a geometric progression.

(a) Prove that $w = -2u$.

(b) Hence, or otherwise, prove that v, u, w form an arithmetic progression.

17. Let $a, b, c, d \in \mathbb{C}\backslash\{0\}$. Suppose $\dfrac{a}{b} = \dfrac{c}{d}$. Further suppose that a, b, c form an arithmetic progression. Prove that $\dfrac{1}{b}, \dfrac{1}{c}, \dfrac{1}{d}$ form an arithmetic progression.

18. (a)$^\diamondsuit$ Let $k \in \mathbb{N}\backslash\{0, 1\}$. Suppose $b_0, b_1, b_2, \cdots, b_k$ form arithmetic progression. Further suppose $b_0 + b_1 + b_2 + \cdots + b_k = 0$. Prove that $b_j + b_{k-j} = 0$ for each $j \in [\![0, k]\!]$.

(b) Let $\{c_p\}_{p=0}^{\infty}$ be an arithmetic progression.
Let $m, n \in \mathbb{N}$. Suppose $m < n$. Suppose $c_0 + c_1 + c_2 + \cdots + c_m = c_0 + c_1 + c_2 + \cdots + c_n$.
Prove that $c_0 + c_1 + c_2 + \cdots + c_{m+n+1} = 0$.

19. Let $x_1, x_2, \cdots, x_n$ be numbers.

(a) Prove that $\displaystyle\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} x_{n+1-k}$.

(b) Prove that $\displaystyle\sum_{k=1}^{n} (x_k + x_{n+1-k}) = 2\sum_{k=1}^{n} x_k$.

(c) Prove that $\displaystyle\sum_{k=1}^{n}\sum_{j=1}^{n} (x_k + x_j) = 2n\sum_{k=1}^{n} x_k$.

20. Let p be a positive number. Let q be a positive integer.

(a) Prove that $\displaystyle\sum_{k=1}^{q} \frac{1}{p+k} = \sum_{j=0}^{q-1} \frac{1}{p+q-j}$.

(b) Hence, or otherwise, deduce that $\displaystyle\sum_{k=0}^{q-1}\frac{1}{p-q+k}+\sum_{k=1}^{q}\frac{1}{p+k}=2p\sum_{k=0}^{q-1}\frac{1}{p^2-(q-k)^2}$.

21. Let n be a positive integer. Define $\displaystyle A=\prod_{k=1}^{n}\left(1-\frac{1}{2k+1}\right)$, $\displaystyle B=\prod_{k=1}^{n}\left(1-\frac{1}{2k}\right)$ and $\displaystyle C=\prod_{k=1}^{n}\left(1+\frac{1}{4k^2-1}\right)$.

(a) Prove that $A=BC$.

(b) Prove that $\displaystyle C=\frac{P^n(n!)^Q}{[(2n)!][(2n+1)!]}$. Here P, Q are integers whose respective values you have to determine explicitly.

22. Let n be a positive integer. Prove that $\displaystyle\prod_{k=1}^{n}\left(\frac{2n-2k+1}{2n+2k-1}\right)=\frac{[(An)!]^B}{[(A^2n)!](n!)^C}$. Here A, B, C are integers whose respective values you have to determine explicitly.