MATH1050 Exercise 1 (Answers and selected solution)

1. **Solution.**

(a) We proceed to solve the equation $(\star)$:

$$\begin{array}{rcl} x+\sqrt{x+1} & = & 11 \quad \text{——} \quad (\star) \\ \sqrt{x+1} & = & 11-x \\ (\sqrt{x+1})^2 & = & (11-x)^2 \\ x+1 & = & x^2-22x+121 \\ x^2-23x+120 & = & 0 \\ (x-8)(x-15) & = & 0 \\ x=8 & \text{or} & x=15 \end{array}$$

Checking:

- $8+\sqrt{8+1}=11$.
- $15+\sqrt{15+1}=19\neq 11$.

The only solution of $(\star)$ is $x=8$.

(b) We proceed to solve the equation $(\star)$:

$$\begin{array}{rcl} 2(4^x+4^{-x})-7(2^x+2^{-x})+10 & = & 0 \quad \text{——} \quad (\star) \\ 2(2^x+2^{-x})^2-7(2^x+2^{-x})+6 & = & 0 \\ [2(2^x+2^{-x})-3][(2^x+2^{-x})-2] & = & 0 \\ 2^x-\frac{3}{2}+2^{-x}=0 & \text{or} & 2^x-2+2^{-x}=0 \\ 2^{2x}-\frac{3}{2}\cdot 2^x+1=0 & \text{or} & 2^{2x}-2\cdot 2^x+1=0 \\ \underbrace{(2^x-\frac{3}{4})^2=-\frac{7}{16}}_{\text{(rejected)}} & \text{or} & (2^x-1)^2=0 \\ 2^x & = & 1 \\ x & = & 0 \end{array}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The only solution of $(\star)$ is $x=0$.

(c) We proceed to solve the equation $(\star)$:

$$\begin{array}{rcl} \log_{5-x}(215-x^3) & = & 3 \quad \text{——} \quad (\star) \\ \frac{\ln(215-x^3)}{\ln(5-x)} & = & 3 \\ \ln(215-x^3) & = & 3\ln(5-x) \\ 215-x^3 & = & (5-x)^3 \\ x^2-5x-6 & = & 0 \\ (x+1)(x-6) & = & 0 \\ x=-1 & \text{or} & x=6 \end{array}$$

Checking:

- $5-(-1)=6>0$ and $215-(-1)^3=216>0$. We have $\log_{5-(-1)}(215-(-1)^3)=\log_6(216)=3$.
- $5-6<0$. Then $\log_{5-6}(u)$ is not well-defined for whatever real value of u.

The only solution of $(\star)$ is $x=-1$.

(d) We proceed to solve the equation ($\star$):

$$\begin{aligned}
|x^2-5x+6| &= x \quad \text{——} \quad (\star) \\
x^2-5x+6=x \quad &\text{or} \quad -(x^2-5x+6)=x \\
x^2-6x+6=0 \quad &\text{or} \quad x^2-4x+6=0 \\
(x-3)^2-3=0 \quad &\text{or} \quad \underbrace{(x-2)^2=-2}_{\text{rejected}} \\
(x-3+\sqrt{3})(x-3-\sqrt{3}) &= 0 \\
x=3-\sqrt{3} \quad &\text{or} \quad x=3+\sqrt{3}
\end{aligned}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solution of ($\star$) is $x=3-\sqrt{3}$ or $x=3+\sqrt{3}$.

(e) We proceed to solve the equation ($\star$):

$$\begin{aligned}
x|x|+5x+6 &= 0 \quad \text{——} \quad (\star) \\
x^2+5x+6=0 \quad &\text{or} \quad -x^2+5x+6=0 \\
(x+2)(x+3)=0 \quad &\text{or} \quad -(x+1)(x-6)=0 \\
x=-2 \quad \text{or} \quad x=-3 \quad &\text{or} \quad x=-1 \quad \text{or} \quad x=6
\end{aligned}$$

Checking:

- $(-2)|-2|+5(-2)+6=-8\neq 0.$
- $(-3)|-3|+5(-3)+6=-18\neq 0.$
- $(-1)|-1|+5(-1)+6=0.$
- $6|6|+5\cdot 6+6=72\neq 0.$

The only solution of ($\star$) is $x=-1$.

(f) We proceed to solve the equation ($\star$):

$$\begin{aligned}
(x-4)^2-5|x-4|+6 &= 0 \quad \text{——} \quad (\star) \\
|x-4|^2-5|x-4|+6 &= 0 \\
(|x-4|-2)(|x-4|-3) &= 0 \\
|x-4|=2 \quad &\text{or} \quad |x-4|=3 \\
x-4=2 \quad \text{or} \quad x-4=-2 \quad &\text{or} \quad x-4=3 \quad \text{or} \quad x-4=-3 \\
x=6 \quad \text{or} \quad x=2 \quad &\text{or} \quad x=7 \quad \text{or} \quad x=1
\end{aligned}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solution of ($\star$) is $x=1$ or $x=2$ or $x=6$ or $x=7$.

(g) We proceed to solve the system of equations ($\star$):

$$\left\{\begin{array}{ccccc} xy & + & x & = & 6 \\ xy & - & y & = & 2 \end{array}\right.$$

$$\begin{aligned}
xy+x &= 6 \quad \text{——} \quad (\star_1) \\
xy-y &= 2 \quad \text{——} \quad (\star_2) \\
x+y &= 4 \quad \text{(by ‘subtracting } (\star_2) \text{ from } (\star_1)\text{’)} \\
y &= 4-x \quad \text{——} \quad (\star_3) \\
x(4-x)+x &= 6 \quad \text{(by ‘substituting } (\star_3) \text{ into } (\star_1)\text{’)} \\
x^2-5x+6 &= 0 \\
(x-2)(x-3) &= 0 \\
x=2 \quad &\text{or} \quad x=3
\end{aligned}$$

When $x=2$, $y=4-x=2$.
When $x=3$, $y=4-x=1$.
Checking:

- Suppose $x = 2$ and $y = 2$. Then $xy + x = 2 \cdot 2 + 2 = 6$. Also, $xy - y = 2 \cdot 2 - 2 = 2$.
- Suppose $x = 3$ and $y = 1$. Then $xy + 2 = 3 \cdot 1 + 3 = 6$. Also, $xy - y = 3 \cdot 1 - 1 = 2$.

Hence the solution of the system $(\star)$ is given by $(x, y) = (2, 2)$ or $(x, y) = (3, 1)$.

(h) We proceed to solve the system of equations $(\star)$:

$$\begin{cases} xy &= 35 \\ x^{\log_5(y)} &= 7 \end{cases}$$

$$\begin{aligned}
xy &= 35 \quad \text{——} \quad (\star_1) \\
x^{\log_5(y)} &= 7 \quad \text{——} \quad (\star_2) \\
\log_5(x) + \log_5(y) &= \log_5(7) + 1 \quad \text{——} \quad (\star_3), \quad \text{(by `taking logarithm in } (\star_1)\text{')} \\
\log_5(x) \cdot \log_5(y) &= \log_5(7) \quad \text{——} \quad (\star_4), \quad \text{(by `taking logarithm in } (\star_2)\text{')} \\
\log_5(x) + \log_5(y) - \log_5(x) \cdot \log_5(y) &= 1 \quad \text{(by `subtracting } (\star_4) \text{ from } (\star_3)\text{')} \\
\log_5(x) \cdot \log_5(y) - \log_5(x) - \log_5(y) + 1 &= 0 \\
(\log_5(x) - 1)(\log_5(y) - 1) &= 0 \\
\log_5(x) = 1 \quad &\text{or} \quad \log_5(y) = 1 \\
x = 5 \quad &\text{or} \quad y = 5.
\end{aligned}$$

When $x = 5$, $y = 35/x = 7$.

When $y = 5$, $x = 35/y = 7$.

Checking:

- Suppose $x = 5$ and $y = 7$. Then $xy = 5 \cdot 7 = 35$. Also, $x^{\log_5(y)} = 5^{\log_5(7)} = 7$.
- Suppose $x = 7$ and $y = 5$. Then $xy = 7 \cdot 5 = 35$. Also, $x^{\log_5(y)} = 7^{\log_5(5)} = 7$.

Hence the solution of the system $(\star)$ is given by $(x, y) = (5, 7)$ or $(x, y) = (7, 5)$.

2. **Solution.**

Let c be a real number. Consider the equation

$$\ln(x + c) = \ln(c) + \ln(x) \quad \text{——} \quad (\star_c)$$

with unknown x.

(a) Suppose $(\star_c)$ has a real solution, say, $x = \alpha$. Then

$$\begin{aligned}
\ln(\alpha + c) &= \ln(c) + \ln(\alpha) \\
\ln(\alpha + c) &= \ln(c\alpha) \\
\alpha + c &= c\alpha \\
(c - 1)\alpha &= c \\
\alpha &= \frac{c}{c - 1}
\end{aligned}$$

Checking:

- We have $\ln\left(\frac{c}{c-1} + c\right) - \ln(c) - \ln\left(\frac{c}{c-1}\right) = \cdots = 0.$

  Then $\ln\left(\frac{c}{c-1} + c\right) = \ln(c) + \ln\left(\frac{c}{c-1}\right)$.

Hence $(\star_c)$ has exactly one real solution, namely $x = \frac{c}{c-1}$.

(b) We observe:

- Suppose $c \leq 0$. Then $\ln(c)$ is not well-defined.
- Suppose $0 < c < 1$. Then we have $\frac{c}{c-1} < 0$. Therefore $\ln\left(\frac{c}{c-1}\right)$ is not well-defined.

- Suppose $c = 1$. Then $\dfrac{c}{c-1}$ is not well-defined.
- Suppose $c > 1$. Then we have $\dfrac{c}{c-1} > 0$. Therefore $\ln\left(\dfrac{c}{c-1}\right)$ is well-defined.

  Also, $\ln(c)$ is well-defined.

  Moreover, we have $c + \dfrac{c}{c-1} > 0$. Then $\ln\left(\dfrac{c}{c-1} + c\right)$ is well-defined.

Hence $(\star_c)$ has a real solution iff $c > 1$.

3. **Answer.**

(a) (I) There exist some

(II) $n \neq 0$ and

(III) there exist some $p, q \in \mathbb{Z}$ such that

(IV) $n \neq 0$

(V) $q \neq 0$

(VI) $mq + pn \in \mathbb{Z}$ and $nq \in \mathbb{Z}$

(b) (I) There exist some $m, n \in \mathbb{Z}$ such that

(II) there exists some $p, q \in \mathbb{Z}$ such that $q \neq 0$ and $p = qy$

(III) $mp = nxqy = nq(xy)$

(IV) $nq \neq 0$

(V) since $m, n, p, q \in \mathbb{Z}$, we have, $mp \in \mathbb{Z}$ and $nq \in \mathbb{Z}$

(c) i. *Hint.* Modify the proof for the statement (S).

ii. *Hint.* Modify the proof for the statement (P).

4. (a) **Answer.**

DGECBFA.

Alternative answers. **DGCEBFA**, **DGECFBA**, **DGCEFBA**.

(b) **Answer.**

(I) Let $x, y, n \in \mathbb{Z}$. Suppose x is divisible by n or y is divisible by n.

(II) there exists some $k \in \mathbb{Z}$ such that $x = kn$

(III) $xy = (kn)y = (ky)n$

(IV) since $k \in \mathbb{Z}$ and $y \in \mathbb{Z}$, we have $ky \in \mathbb{Z}$

(V) xy is divisible by n

(VI) in any case, xy is divisible by n

Remark. The entries for (III), (IV) may be interchanged.

(c) ——

5. **Solution.**

Suppose $a_0, a_1, a_2, \cdots$ are in geometric progression, with common ratio r. Suppose $m, n, p \in \mathbb{N}$, and $a_m = A$, $a_n = B$ and $a_p = C$.

Then $a_m = a_0 r^m$, $a_n = a_0 r^n$ and $a_p = a_0 r^p$.

Therefore

$$\begin{aligned} A^{n-p}B^{p-m}C^{m-n} &= (a_0 r^m)^{n-p}(a_0 r^n)^{p-m}(a_0 r^p)^{m-n} \\ &= {a_0}^{(n-p)+(p-m)+(m-n)} \cdot r^{m(n-p)+n(p-m)+p(m-n)} = {a_0}^0 r^0 = 1. \end{aligned}$$

6. **Solution.**

Let a, b, c be numbers.

(a) Suppose a, b, c are in arithmetic progression. Denote the common diffference by d. We have $b - a = d$ and $c - b = d$.
Note that $[(b^2 - ca) - (a^2 - bc)] = b^2 - a^2 - ca + bc = (b - a)(b + a) + (b - a)c = d(a + b + c)$.
Also note that $[(c^2 - ab) - (b^2 - ca)] = c^2 - b^2 - ab + ca = (c - b)(c + b) + (c - b)a = d(a + b + c)$.
Then $(b^2 - ca) - (a^2 - bc) = (c^2 - ab) - (b^2 - ca)$.
Hence $a^2 - bc, b^2 - ca, c^2 - ab$ are in arithmetic progression.

(b) Suppose $a^2 - bc, b^2 - ca, c^2 - ab$ are in arithmetic progression. Further suppose $a + b + c \neq 0$.
Then $b^2 - ca = \dfrac{(a^2 - bc) + (c^2 - ab)}{2}$.
Therefore $2b^2 - 2ca = a^2 + c^2 - ab - bc$.
Hence $0 = (a^2 - b^2) + (c^2 - b^2) + (ac - ab) + (ac - bc) = \cdots = (a + c - 2b)(a + b + c)$.
Since $a + b + c \neq 0$, we have $a + c - 2b = 0$. Then $b = \dfrac{a+b}{2}$.
Therefore a, b, c are in arithmetic progression.

7. (a) **Solution.**
Let r be any number not equal to 1.

i. For each $n \in \mathbb{N}$, write $T_n(r) = 1 + r + r^2 + \cdots + r^{n-1} + r^n$.
By definition, $rT_n(r) = r + r^2 + r^3 \cdots + r^n + r^{n+1}$.
Therefore $(1 - r)T_n(r) = T_n(r) - rT_n(r) = 1 - r^{n+1}$.
Since $r \neq 1$, we have $T_n(r) = \dfrac{1 - r^{n+1}}{1 - r}$.

ii. Now suppose $r \neq 0$ as well. Since $r \neq 1$, we have $\dfrac{1}{r} \neq 1$.
For each $n \in \mathbb{N}$, write $T_n(r) = 1 + r + r^2 + \cdots + r^{n-1} + r^n$.
Then $T_n(\frac{1}{r}) = \dfrac{1 - (1/r)^{n+1}}{1 - 1/r} = \dfrac{1 - r^{n+1}}{r^n(1 - r)}$. Therefore

$$\begin{aligned} r^n + r^{n-1} + \cdots + r + 1 + \frac{1}{r} + \cdots + \frac{1}{r^{n-1}} + \frac{1}{r^n} &= T_n(r) + T_n(\frac{1}{r}) - 1 \\ &= \frac{1 - r^{n+1}}{1 - r} + \frac{1 - r^{n+1}}{r^n(1 - r)} - 1 \\ &= \frac{r^n(1 - r^{n+1}) + (1 - r^{n+1}) - r^n(1 - r)}{r^n(1 - r)} \\ &= \frac{1 - r^{2n+1}}{r^n(1 - r)} \end{aligned}$$

(b) **Answer.**
$A = B = 1$, $C = D = 2$.
Hint. Use the same trick for part (a). Start by writing $U_m(s) = 1 + 2s + 3s^2 + \cdots + ms^{m-1} + (m + 1)s^m$. Then study the expression $U_m(s) - sU_m(s)$.

(c) **Solution.**
Let $n \in \mathbb{N}$. Suppose a, b be any numbers.

i. Suppose $a = b$.
Then $a^{n+1} - b^{n+1} = 0 = (a - b)(a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + a^2b^{n-2} + ab^{n-1} + b^n)$.
Suppose $a \neq b$. Then a, b are not both zero. Without loss of generality, suppose $a \neq 0$. Write $r = \dfrac{b}{a}$
Then

$$\begin{aligned} a^{n+1} - b^{n+1} &= a^{n+1}\left[1 - \left(\frac{b}{a}\right)^{n+1}\right] = a^{n+1}(1 - r^{n+1}) \\ &= a^{n+1}(1 - r)(1 + r + r^2 + \cdots + r^n) \\ &= a(1 - r) \cdot a^n(1 + r + r^2 + \cdots + r^n) \\ &= (a - b)(a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + a^2b^{n-2} + ab^{n-1} + b^n) \end{aligned}$$

ii. Modifying the argument in the previous part, we obtain $(a-b)^2[a^m + 2a^{m-1}b + 3a^{m-2}b^2 + \cdots + mab^{m-1} + (m+1)b^m] = a^{m+2} - (m+2)ab^{m+1} + (m+1)b^{m+2}$.

8. ——

9. *Hint.* Refer to the definition for the notion of geometric progression. Can we link up each of numbers $c_1, c_2, \cdots, c_n$ with c_0 through introducing a certain number which is not explicitly mentioned in the set-up but is implicitly provided by the notion of geometric progression?