MATH1050 Exercise 1

1. Solve for all real solutions of each of the equations/systems below:

   (a) $x + \sqrt{x+1} = 11$.

   (b) $2(4^x + 4^{-x}) - 7(2^x + 2^{-x}) + 10 = 0$.

   (c) $\log_{5-x}(215 - x^3) = 3$.

   (d) $|x^2 - 5x + 6| = x$.

   (e) $x|x| + 5x + 6 = 0$.

   (f) $(x-4)^2 - 5|x-4| + 6 = 0$.

   (g) $\begin{cases} xy + x = 6 \\ xy - y = 2 \end{cases}$

   (h) $\begin{cases} xy = 35 \\ x^{\log_5(y)} = 7 \end{cases}$

2. Let $c$ be a real number. Consider the equation

$$\ln(x+c) = \ln(c) + \ln(x) \quad \text{——} \quad (\star_c)$$

   with unknown $x$.

   (a) Suppose $(\star_c)$ has a real solution. Find all real solutions of $(\star_c)$.

   (b)$^\diamond$ For which values of $c$ does $(\star_c)$ have any real solution? Justify your answer.

3. We introduce the following definition:

   - *Let $x \in \mathbb{R}$. $x$ is said to be* **rational** *if there exist some $m, n \in \mathbb{Z}$ such that $m = nx$ and $n \neq 0$.*

   (a) Consider the statement $(S)$:

   $(S)$ *Let $x, y$ be rational numbers. $x + y$ is a rational number.*

   Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement $(S)$. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   > Let $x, y$ be rational numbers.
   > ___(I)___ $m, n \in \mathbb{Z}$ such that ___(II)___ $m = nx$. Also, ___(III)___ $q \neq 0$ and $p = qy$.
   > Note that $mq + pn = nxq + qyn = nq(x+y)$ .
   > Since ___(IV)___ and ___(V)___ , we have $nq \neq 0$.
   > Also, since $m, n, p, q \in \mathbb{Z}$, we have ___(VI)___ .
   > Hence $x + y$ is a rational number.

   (b) Consider the statement $(P)$:

   $(P)$ *Let $x, y$ be rational numbers. $xy$ is a rational number.*

   Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement $(P)$. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   > Let $x, y$ be rational numbers.
   > ___(I)___ $n \neq 0$ and $m = nx$. Also, ___(II)___ .
   > Note that ___(III)___ .
   > Since $n \neq 0$ and $q \neq 0$, we have ___(IV)___ .
   > Also, ___(V)___ .
   > Hence $xy$ is a rational number.

(c) Prove the statements below:

i. *Let $x, y$ be rational numbers. $x - y$ is a rational number.*

ii. *Let $x, y$ be rational numbers. Suppose $y \neq 0$. Then $\dfrac{x}{y}$ is a rational number.*

4. (a) Consider the statement ($T$):

($T$) *Let $x, y, n \in \mathbb{Z}$. Suppose $x$ is divisible by $n$ and $y$ is divisible by $n$. Then $x + y$ is divisible by $n$.*

By an appropriate re-ordering of the blocks of sentences in the box below, labelled by bold-typed Latin alphabets **A**, **B**, ..., **G** respectively, give a proof for the statement ($T$):

> **A**. *Then, by definition, $x + y$ is divisible by $n$.*
>
> **B**. *Note that $x + y = kn + \ell n = (k + \ell)n$.*
>
> **C**. *Since $y$ is divisible by $n$, there exists some $\ell \in \mathbb{Z}$ such that $y = \ell n$.*
>
> **D**. *Let $x, y, n \in \mathbb{Z}$.*
>
> **E**. *Since $x$ is divisible by $n$, there exists some $k \in \mathbb{Z}$ such that $x = kn$.*
>
> **F**. *Since $k \in \mathbb{Z}$ and $\ell \in \mathbb{Z}$, we have $k + \ell \in \mathbb{Z}$.*
>
> **G**. *Suppose $x$ is divisible by $n$ and $y$ is divisible by $n$.*

(b) Consider the statement ($V$):

($V$) *Let $x, y, n \in \mathbb{Z}$. Suppose $x$ is divisible by $n$ or $y$ is divisible by $n$. Then $xy$ is divisible by $n$.*

Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement ($V$). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

> ________(I)________
>
> - (Case 1). Suppose $x$ is divisible by $n$. Then _____(II)_____ .
>   Note that _____(III)_____ . Also, _____(IV)_____ .
>   Then _____(V)_____ .
> - (Case 2). Suppose $y$ is divisible by $n$. Modifying the argument for (Case 1), we also deduce that $xy$ is divisible by $n$.
>
> Hence, _____(VI)_____ .

(c) Give a 'direct proof' for the statement below:

- *Let $x, y \in \mathbb{Z}$. Suppose $x$ is divisible by $y$ and $y$ is divisible by $x$. Then $|x| = |y|$.*

5. Suppose $a_0, a_1, a_2, \cdots$ are in geometric progression, with common ratio $r$. Suppose $m, n, p \in \mathbb{N}$, and $a_m = A$, $a_n = B$ and $a_p = C$. Prove that $A^{n-p}B^{p-m}C^{m-n} = 1$.

6. Let $a, b, c$ be numbers.

(a) Suppose $a, b, c$ are in arithmetic progression. Prove that $a^2 - bc, b^2 - ca, c^2 - ab$ are in arithmetic progression.

(b) Suppose $a^2 - bc, b^2 - ca, c^2 - ab$ are in arithmetic progression. Further suppose $a + b + c \neq 0$. Prove that $a, b, c$ are in arithmetic progression.

7. (a) Let $n \in \mathbb{N}$. Let $r$ be any number which is not equal to 1.

i. Verify that $1 + r + r^2 + \cdots + r^{n-1} + r^n = \dfrac{1 - r^{n+1}}{1 - r}$.

(Hint: Start by writing $T(r) = 1 + r + r^2 + \cdots + r^{n-1} + r^n$. Then study the expression $T(r) - rT(r)$.)

ii. Further suppose $r \neq 0$. Prove that $r^n + r^{n-1} + \cdots + r + 1 + \frac{1}{r} + \cdots + \frac{1}{r^{n-1}} + \frac{1}{r^n} = \frac{1 - r^{2n+1}}{r^n(1-r)}$.

(b)$^\diamond$ Let $m \in \mathbb{N}$. Let $s$ be any number which is not equal to 1. Verify that

$$1 + 2s + 3s^2 + \cdots + ms^{m-1} + (m+1)s^m = \frac{A - (m+2)s^{m+B} + (m+1)s^{m+C}}{(1-s)^D},$$

where $A, B, C, D$ are integers independent of the value of $s$. You have to determine the respective values of $A, B, C, D$ explicitly.

(c) Let $n \in \mathbb{N}$. Let $a, b$ be any real numbers.

i. Prove that $a^{n+1} - b^{n+1} = (a-b)(a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + a^2b^{n-2} + ab^{n-1} + b^n)$.
(Hint: Consider the cases '$a = b$', '$a \neq b$' separately. Make use of part (a) for the case '$a \neq b$'. )

ii.$^\diamond$ Prove that $(a-b)^E[a^m + 2a^{m-1}b + 3a^{m-2}b^2 + \cdots + mab^{m-1} + (m+1)b^m] = a^{m+F} - (m+G)ab^{m+H} + (m+J)b^{m+K}$. Here $E, F, G, H, J, K$ are integers independent of the value of $m$. You have to determine the respective values of $E, F, G, H, J, K$ explicitly.

8. Let $n$ be a positive integer, and $x_1, x_2, \cdots, x_n$ be real numbers. Define $\bar{x} = \frac{1}{n}\sum_{k=1}^{n} x_k$.

(a) i. Prove that $\sum_{j=1}^{n}(x_j - \bar{x}) = 0$.

ii. Prove that $\sum_{j=1}^{n}(x_j - \bar{x})^2 = \sum_{j=1}^{n} {x_j}^2 - n\bar{x}^2$.

(b) Let $a, b$ be real numbers, with $a \neq 0$. For each $j = 1, 2, \cdots, n$, define $y_j = ax_j + b$. Define $\bar{y} = \frac{1}{n}\sum_{k=1}^{n} y_k$.

i. Prove that $\bar{y} = a\bar{x} + b$.

ii. Prove that $\sum_{j=1}^{n}(y_j - \bar{y})^2 = a^2\left(\sum_{j=1}^{n} {x_j}^2 - n\bar{x}^2\right)$.

9.$^\diamond$ Let $c_0, c_1, c_2, \cdots, c_n \in \mathbb{C}\backslash\{0\}$. Suppose $c_0, c_1, c_2, \cdots, c_n$ form a geometric progression.

Define $S = c_0 + c_1 + c_2 + \cdots + c_n$, $P = c_0 \cdot c_1 \cdot c_2 \cdot ... \cdot c_n$, and $R = \frac{1}{c_0} + \frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_n}$.

Prove that $\left(\frac{S}{R}\right)^{n+1} = P^2$.