MATH1050 Construction of the integer system from the natural number system

1. We assume all the basic properties of the natural number system, which is made up of the set $\mathbb{N}$ and its addition, its multiplication and its usual ordering. In the natural number system, we may perform addition and multiplication freely. The same cannot be said of subtraction: we cannot subtract a larger natural from a smaller one within the natural number system.

   In the procedure outlined below, we are going to construct the integer system, in which we may preform subtraction freely. From the philosophical standpoint, the integer system is something which we do not have at this moment: we are going to 'create' the integer system from the natural number system, so that we may perforem subtraction freely.

2. **Theorem (1).**

   Define the relation $R_Z = (\mathbb{N}^2, \mathbb{N}^2, E_Z)$ by

   $$E_Z = \{((x, y), (s, t)) \mid x, y, s, t \in \mathbb{N} \text{ and } x + t = s + y\}.$$

   R_Z is an equivalence relation.

   The equivalence classes under R_Z are the desired integers, and the quotient $\mathbb{Z}$ of $\mathbb{N}^2$ by R_Z is the desired set of all integers.

   Remark. For each pair of natural numbers (x, y), the equivalence class $[(x, y)]$ is given by

   $$[(x, y)] = \{..., (x - 2, y - 2), (x - 1, y - 1), (x, y), (x + 1, y + 1), (x + 2, y + 2), ...\}.$$

   - Suppose $x > y$. Write $z = x - y$. Then $[(x, y)] = \{(z, 0), (z + 1, 1), (z + 2, 2), ...\}$: this integer is what we desire to be the positive integer z.
   - Suppose $x < y$. Write $z' = y - x$. Then $[(x, y)] = \{(0, z'), (1, z' + 1), (2, z' + 2)...\}$: this integer is what we desire to be the negative integer $-z'$.
   - Suppose $x = y$. Then $[(x, y)] = \{(0, 0), (1, 1), (2, 2), ...\}$: this integer is what we desire to be the integer 0.

3. **Theorem (2).**

   Define the relation $\alpha = (\mathbb{Z}^2, \mathbb{Z}, G_\alpha)$ by

   $$G_\alpha = \left\{ ((u, v), w) \;\middle|\; \begin{array}{l} \text{There exist } k, \ell, m, n \in \mathbb{N} \text{ such that} \\ u = [(k, \ell)], v = [(m, n)] \text{ and } w = [(k + m, \ell + n)] \end{array} \right\}.$$

   α is a function.

   This function α the addition in $\mathbb{Z}$ that we hope for. From now on we write $\alpha(u, v)$ as $u + v$, and call it the sum of u, v.

4. **Theorem (3).**

   The statements below hold:

   (a) *For any $u, v \in \mathbb{Z}$, $u + v \in \mathbb{Z}$.*

   (b) *For any $u, v \in \mathbb{Z}$, $u + v = v + u$.*

   (c) *For any $u, v, w \in \mathbb{Z}$, $(u + v) + w = u + (v + w)$.*

   (d) *For any $u \in \mathbb{Z}$, $[(0, 0)] + u = u + [(0, 0)] = u$.*

   (e) *For any $u \in \mathbb{Z}$, there exists some $v \in \mathbb{Z}$ such that $u + v = v + u = [(0, 0)]$.*

   Hence the addition of integers possesses the expected basic properties.

   Corollary (4).

   $(\mathbb{Z}, +)$ is an abelian group.

5. **Theorem (5).**

   The statement below holds:

   (S_Z) *For any $u, v \in \mathbb{Z}$, there exists some unique $w \in \mathbb{Z}$ such that $u + w = v$.*

   Hence subtraction can be performed freely amongst integers.

   From now on, whenever $u + w = v$, we may write $w = v - u$. We call $v - u$ the difference of v from u. We write $[(0, 0)]$ as $0_{\mathbb{Z}}$, and call it the integer zero. We write $0_{\mathbb{Z}} - u$ as $-u$.

6. *Theorem (6).*

   Define the relation $\mu = (\mathbb{Z}^2, \mathbb{Z}, G_\mu)$ *by*

$$G_\mu = \left\{ ((u,v),w) \;\middle|\; \begin{array}{l} \text{There exist } k,\ell,m,n \in \mathbb{N} \text{ such that} \\ u = [(k,\ell)], v = [(m,n)] \text{ and } w = [(km+\ell n, kn+\ell m)] \end{array} \right\}.$$

   μ is a function.

   This function μ is the multiplication in $\mathbb{Z}$ that we hope for. From now on we write $\mu(u,v)$ as $u \times v$, and call it the product of u, v.

7. **Theorem (7).**

   The statements below hold:

   (a) *For any* $u, v \in \mathbb{Z}$, $u \times v \in \mathbb{Z}$.

   (b) *For any* $u, v \in \mathbb{Z}$, $u \times v = v \times u$.

   (c) *For any* $u, v, w \in \mathbb{Z}$, $(u \times v) \times w = u \times (v \times w)$.

   (d) *For any* $u, v, w \in \mathbb{Z}$, $u \times (v+w) = (u \times v) + (u \times w)$ *and* $(u+v) \times w = (u \times w) + (v \times w)$.

   (e) *For any* $u \in \mathbb{Z}$, $u \times [(1,0)] = [(1,0)] \times u = u$.

   (f) *For any* $u, v \in \mathbb{Z}$, *if* $u \times v = 0_{\mathbb{Z}}$ *then* $u = 0_{\mathbb{Z}}$ *or* $v = 0_{\mathbb{Z}}$.

   Hence the multiplication of integers possesses the expected basic properties. From now we write $[(1,0)]$ as $1_{\mathbb{Z}}$, and call it the integer one.

   Corollary (8).

   $(\mathbb{Z}, +, \times)$ *is an integral domain.*

8. **Theorem (9).**

   Define the relation $P = (\mathbb{Z}, \mathbb{Z}, G_P)$ *by*

$$G_P = \left\{ (u,v) \;\middle|\; \begin{array}{l} \text{There exist some } k,\ell,m,n \in \mathbb{N} \text{ such that} \\ u = [(k,\ell)], v = [(m,n)] \text{ and } k+n \leq \ell+m \end{array} \right\}.$$

   P is a total ordering in $\mathbb{Z}$.

   This P is the 'usual ordering for integers' that we hope for.

   From now on, we write $u \leq v$ (or equivalently $v \geq u$) exactly when $(u,v) \in P$,.

9. **Theorem (10).**

   The statements below hold:

   (a) *For any* $x, y \in \mathbb{N}$, $[(x,y)] \geq 0_{\mathbb{Z}}$ *iff* $x \geq y$.

   (b) *For any* $x, y \in \mathbb{N}$, $[(x,y)] \leq 0_{\mathbb{Z}}$ *iff* $x \leq y$.

   (c) *For any* $u, v \in \mathbb{Z}$, *if* $u \geq 0_{\mathbb{Z}}$ *and* $v \geq 0_{\mathbb{Z}}$ *then* $u + v \geq 0_{\mathbb{Z}}$ *and* $u \times v \geq 0_{\mathbb{Z}}$.

   Hence the usual ordering for integers possesses its expected basic properties.

   From now on, whenever $u \geq 0_{\mathbb{Z}}$, we call u a non-negative integer. Whenever $u \geq 0_{\mathbb{Z}}$ and $u \neq 0_{\mathbb{Z}}$, we write $u > 0_{\mathbb{Z}}$ and call u a positive integer. Whenever $u \leq 0_{\mathbb{Z}}$, we call u a non-positive integer. Whenever $u \leq 0_{\mathbb{Z}}$ and $u \neq 0_{\mathbb{Z}}$, we write $u < 0_{\mathbb{Z}}$ and call u a negative integer.

10. **Theorem (11).**

    Define the function $\iota : \mathbb{N} \longrightarrow \mathbb{Z}$ *by* $\iota(n) = [(n,0)]$ *for any* $n \in \mathbb{N}$.

    The statements below hold:

    (a) *ι is an injective function.*

    (b) $\iota(\mathbb{N}) = \{u \in \mathbb{Z} \mid u \geq 0_{\mathbb{Z}}\}$.

    (c) *For any* $x, y \in \mathbb{N}$, $\iota(x+y) = \iota(x) + \iota(y)$.

    (d) *For any* $x, y \in \mathbb{N}$, $\iota(xy) = \iota(x) \times \iota(y)$.

    (e) *For any* $x, y \in \mathbb{N}$, $(x \leq y$ *iff* $\iota(x) \leq \iota(y))$.

    From now on we identify $\mathbb{N}$ with $\iota(\mathbb{N})$, which we call the set of all non-negative integers. We write $0_{\mathbb{Z}} = 0$, and write $[(x,0)] = x$, $[(0,y)] = -y$ for each $x, y \in \mathbb{N}$. We identify respectively the addition, the multiplication, the usual ordering in the natural number system and the integer system as each other. The natural number system is now recovered as a subsystem of the integer system which we have constructed above.