1. **Question**.

   - How do we visualize the two most 'basic' number systems, the natural number system and the real number system?

   Answer.

   - Natural number system: (0,) 1, 2, 3, 4, 5, 6, ..., so forth and so on. For comparison of value, we have a 'natural ordering'. For addition, count and count and count; for multiplication, count and add.
   - Real number system: the 'infinite' real line in the 'flat' plane/space, with a reference point which we call 0; points on the line are the numbers. For comparison of value, compare relative positions on the line. For addition and multiplication, do geometry with ruler and compass.

   Question.

   - Are these two points of view compatible? Why bothered? If not compatible, how to reconcile them?

   Answer.

   - We count in terms of natural numbers, and we measure distances in terms of real numbers. We do not 'measure' (natural) numbers. We cannot 'count' distances. So these two points of view are not quite compatible.

     But we need both systems: both counting and measuring are important. They may be needed simultaneously: we often think of the natural number system as a 'subsystem' of the real number system, with the integer system and the rational number system 'in-between' them.

2. **Reconciliation** between the two points of view?

   There are two approaches.

   Approach (1).

   Take everything in the real number system for granted, and regard all others except the complex number system to be 'restrictions from the reals'.

   In your *analysis* course, you will most probably regard the real number system and the above visualization as something more 'fundamental'. You will be given a collection of 'axioms for the real number system'.

   Then you think of the other number systems as subsystems of the real number system; the expected properties of these subsystems have to be 'deduced' from those of the real number system, (according to the 'axioms for the real number system' given to you).

   For example, you will regard the set of all natural numbers as a subset of the set of all real numbers. You will regard addition, multiplication, and the 'usual' ordering for natural numbers as restrictions of the same mathematical objects with these names for the real numbers. You will regard the Well-ordering Principle for Integers and the Principle of Mathematical Induction as some consequences of the axioms for the real number system.

   Advantage of this approach:

   - all subsystems are 'already there', contained in the real number system, waiting to be 'discovered'.

   Disadvantage of this approach:

   - the natural numbers (and the integers) look 'unnatural'; we have to abandon our childhood ideas on the natural numbers for the sake of doing analysis.

   Approach (2).

   Take everything in the natural number system for granted, and regard all others as 'constructions' building up from the natural number system.

   You regard the natural number system and the above visualization as something more 'fundamental'.

   You start the 'axioms for the natural number system' (say, a collection of statements which are logically equivalent to Peano's Axioms). Then you make use of set language (sets, functions, relations) to construct successively the integer system, the rational number system, the real number system, (and if we like, the complex number system as well).

   The expected properties of these super-systems have to be deduced. Moreover, at each step, you identify the old system on which you build up a new system as a subsystem of the new system.

   For example, we construct the integer system from the natural number system. We construct the set of all integers. Then we define addition, multiplication, and the 'usual' ordering for the integers in terms of those mathematical objects with these names for the natural numbers. After the the completion of this construction, we inspect carefully the integer system to identify something within it which we may refer to as the 'natural number subsystem': this amounts to writing down an appropriate injective function from the natural number system to the integer system which preserves addition, multiplication, and 'usual' ordering for the respective systems.

Special feature in this process:

- In each new system constructed, we enable ourselves to do something in the new system that we cannot do in the old system.

Advantage of this approach:

- '*all are numbers.*' (Read it as 'all are natural numbers ultimately'. This is Pythagoras' belief, also Kronecker's.)

Disadvantage of this approach:

- this process of construction is technically involved.

Irrespective of this disadvantage, for this approach to work, we need be able to pinpoint the differences in the respective nature of the various systems, all of which we take for granted in school mathematics.

3. **The natural number system versus the integer system**.

Compare the three statements (S_N°), (S_N), (S_Z):

(S_N°) *For any* $x, y \in \mathbb{N}$, *if* $x \leq y$ *then there exists some unique* $z \in \mathbb{N}$, *namely,* $z = y - x$, *such that* $x + z = y$.

(S_N) *For any* $x, y \in \mathbb{N}$, *there exists some* $z \in \mathbb{N}$ *such that* $x + z = y$.

(S_Z) *For any* $u, v \in \mathbb{Z}$, *there exists some unique* $w \in \mathbb{Z}$, *namely,* $w = v - u$, *such that* $u + w = v$.

(S_N°) is true. (S_N) is false but (S_Z) is true. Subtraction cannot be performed freely amongst natural numbers, but can be performed freely amongst integers.

4. **The integer system versus the rational number system**.

Compare the three statements (D_Z°), (D_Z), (D_Q):

(D_Z°) *For any* $x, y \in \mathbb{Z}\backslash\{0\}$, *if* x *is divisible by* y *then there exists some unique* $z \in \mathbb{Z}\backslash\{0\}$ *such that* $x = zy$.

(D_Z) *For any* $x, y \in \mathbb{Z}\backslash\{0\}$, *there exists some* $z \in \mathbb{Z}\backslash\{0\}$ *such that* $x = zy$.

(D_Q) *For any* $u, v \in \mathbb{Q}\backslash\{0\}$, *there exists some unique* $w \in \mathbb{Q}\backslash\{0\}$ *such that* $u = wv$.

(D_Z°) is true. (D_Z) is false but (D_Q) is true. Division without remainder cannot be performed freely amongst non-zero integers, but can be performed freely amongst non-zero rational numbers.

5. **The rational number system versus the real number system**.

Compare the two statements (L_Q), (L_R):

(L_Q) *Every infinite sequence in* $\mathbb{Q}$ *which is increasing and is bounded above in* $\mathbb{Q}$ *has a limit in* $\mathbb{Q}$.

(L_R) *Every infinite sequence in* $\mathbb{R}$ *which is increasing and is bounded above in* $\mathbb{R}$ *has a limit in* $\mathbb{R}$.

Alternatively, compare the two statements (L'_Q), (L'_R):

(L'_Q) *Every non-empty subset of* $\mathbb{Q}$ *which is bounded above in* $\mathbb{Q}$ *has a least upper bound in* $\mathbb{Q}$.

(L'_R) *Every non-empty subset of* $\mathbb{R}$ *which is bounded above in* $\mathbb{R}$ *has a least upper bound in* $\mathbb{R}$.

(L_Q), (L'_Q) are false but (L_R), (L'_R) are true. For this reason the real number system is (analytically) complete, whereas the rational number system is not.

6. **The real number system versus the complex number system**.

Compare the two statements (P_R), (P_C):

(P_R) *Every non-constant polynomial with real coefficients has a root in* $\mathbb{R}$.

(P_C) *Every non-constant polynomial with complex coefficients has a root in* $\mathbb{C}$.

(P_R) is false but (P_C) is true.

For this reason the complex number system is algebraically complete, whereas the real number system is not.