1. **Definition.**

Let A be a set.

(1) *A is* **countable** *if* $A \lesssim \mathbb{N}$.

(2) *A is said to be* **countably infinite** *if* $A \sim \mathbb{N}$.

(3) *A is said to be* **uncountable** *if A is not countable.*

Basic examples of countably infinite sets.

(a) $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$;

(b) $\mathbb{N}^2$, $\mathbb{N}^3$, $\mathbb{N}^4$, ...

Basic examples of uncountable sets.

(a) $\mathsf{Map}(\mathbb{N}, \{0, 1\})$, $\mathsf{Map}(\mathbb{N}, [\![0, 9]\!])$, $\mathsf{Map}(\mathbb{N}, \mathbb{N})$;

(b) $[0, 1]$, $\mathbb{R}$, $\mathbb{C}$;

(c) $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$, ... ;

(d) $\mathfrak{P}(\mathbb{N}), \mathfrak{P}(\mathfrak{P}(\mathbb{N})), \mathfrak{P}(\mathfrak{P}(\mathfrak{P}(\mathbb{N})))$, ...

Definition.

Let A be a set.

(1) *A is* **countable** *if* $A \lesssim \mathbb{N}$.

(2) *A is said to be* **countably infinite** *if* $A \sim \mathbb{N}$.

(3) *A is said to be* **uncountable** *if A is not countable.*

Theorem (XXI).

(1) *Let A be a set. A is countable iff (A is finite or A is countably infinite).*

(2) *Let A be a set. A is countably infinite iff A is both countable and infinite.*

(3) *A is uncountable iff $\mathbb{N} < A$.*

Remark.

The arguments of the respective statements are word games, possibly involving the application of Schröder-Bernstein Theorem.

Definition.

Let A be a set.

(1) *A is* **countable** *if $A \precsim \mathbb{N}$.*

(2) *A is said to be* **countably infinite** *if $A \sim \mathbb{N}$.*

(3) *A is said to be* **uncountable** *if A is not countable.*

Theorem (XXI).

(1) *Let A be a set. A is countable iff (A is finite or A is countably infinite).*

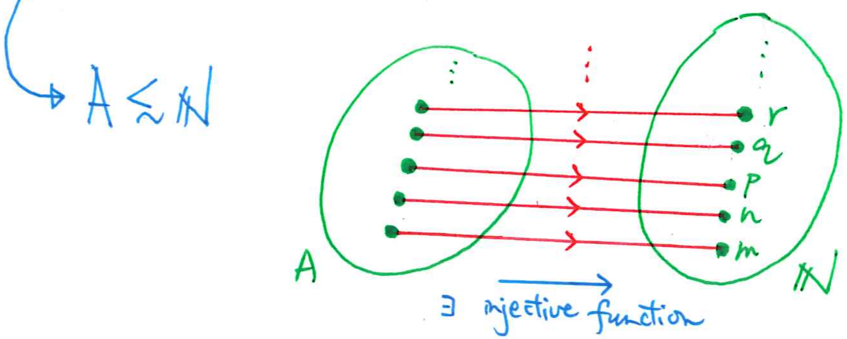
(2) *Let A be a set. A is countably infinite iff A is both countable and infinite.*

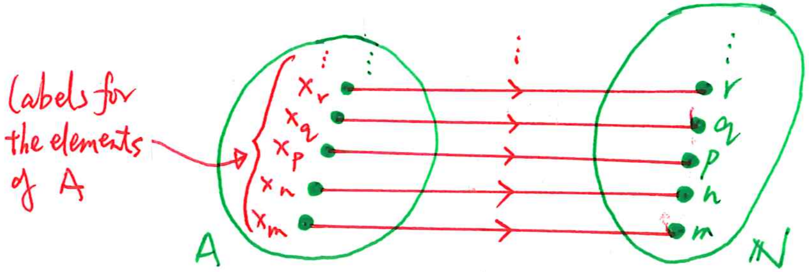
(3) *A is uncountable iff $\mathbb{N} < A$.*

Further remarks.

(a) Heuristic idea on 'being countable':

A is countable exactly when we can identify A as a subset of $\mathbb{N}$ by labeling the elements of A exhaustively by natural numbers.





Definition.

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Theorem (XXI).

(1) *Let A be a set. A is countable iff (A is finite or A is countably infinite).*

(2) *Let A be a set. A is countably infinite iff A is both countable and infinite.*

(3) *A is uncountable iff $\mathbb{N} < A$.*

Further remarks.

(a) ...

(b) Heuristic idea on 'being uncountable':

A is uncountable exactly when there are 'so many' elements in A that there is no way to label all the elements of A by natural numbers alone.

(c) We may think of $\mathbb{N}$ or whatever countably infinite set as a 'smallest' infinite set.

Definition.

Let A be a set.

(1) *A is* **countable** *if $A \lesssim \mathbb{N}$.*

(2) *A is said to be* **countably infinite** *if $A \sim \mathbb{N}$.*

(3) *A is said to be* **uncountable** *if A is not countable.*

Theorem (XXI).

(1) *Let A be a set. A is countable iff (A is finite or A is countably infinite).*

(2) *Let A be a set. A is countably infinite iff A is both countable and infinite.*

(3) *A is uncountable iff $\mathbb{N} < A$.*

Further remarks.

Classification of sets by comparing 'relative sizes' with the 'smallest' infinite set $\mathbb{N}$:

A is finite. ($A < \mathbb{N}$.)	A is infinite. ($\mathbb{N} \lesssim A$.)	*same as left.*
same as above.	A is countably infinite. ($A \sim \mathbb{N}$.)	*same as below.*
same as right.	A is countable. ($A \lesssim \mathbb{N}$.)	A is uncountable. ($\mathbb{N} < A$.)

2. **'Hilbert's Hotel'.**

Every 'infinite' set can 'absorb' a countably infinite set (or a finite set, which is a set of 'smaller size' than $\mathbb{N}$) to form a new set of the 'same size' as itself.

Theorem (XXII). ('Hilbert's Hotel'.)

Let A, B be sets.

Suppose A is infinite and B is countable. Further suppose $A \cap B = \emptyset$.

Then $A \cup B \sim A$.

Proof. The result follows from Theorem (XXI), Lemma (XXIII) and Lemma (XXIV).

Lemma (XXIII).

Let A, B be sets.

Suppose A is infinite and B is finite. Further suppose $A \cap B = \emptyset$.

Then $A \cup B \sim A$.

Lemma (XXIV).

Let A, B be sets.

Suppose A is infinite and B is countably infinite. Further suppose $A \cap B = \emptyset$.

Then $A \cup B \sim A$.

Lemma (XXIII).

Let A, B be sets.

Suppose A is infinite and B is finite. Further suppose $A \cap B = \emptyset$.

Then $A \cup B \sim A$.

Proof of Lemma (XXIII).

Let A, B be sets. Suppose A is infinite and B is finite. Further suppose $A \cap B = \emptyset$.

There is an injective function from $\mathbb{N}$ to A.

This injective function defines some infinite sequence $\{x_n\}_{n=0}^{\infty}$ in A with no repeated terms:

- $x_m \neq x_n$ whenever $m \neq n$.

Write $S = \{x_n \mid n \in \mathbb{N}\}$. ($S$ is the set of all terms of $\{x_n\}_{n=0}^{\infty}$.)

Since B is finite, and $|B| = N$ for some $N \in \mathbb{N}$.

Write $B = \{y_0, y_1, \cdots, y_{N-1}\}$.

Proof of Lemma (XXIII). (*Cont'd.*)

Idea. How to proceed with the construction of a bijective function from A to $A \cup B$?

A	S	x_0	x_1	$\cdots$	x_{N-1}	x_N	x_{N+1}	x_{N+2}	$\cdots$	$z \in A \backslash S$
		$\downarrow$	$\downarrow$	$\cdots$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$	$\downarrow$
$A \cup B$	$S \cup B$	y_0	y_1	$\cdots$	y_{N-1}	x_0	x_{0+1}	x_{0+2}	$\cdots$	$z \in A \backslash S$

Define [with the table above in mind]

$$
\begin{aligned}
F_1 &= \{(x_i, y_i) \mid i \in [\![0, N-1]\!]\} \cup \{(x_j, x_{j-N}) \mid j \in \mathbb{N} \text{ and } j \geq N\}, \\
F_2 &= \{(z, z) \mid z \in A \backslash S\}.
\end{aligned}
$$

Define the relation $f = (A, A \cup B, F)$ by $F = F_1 \cup F_2$.

f is a bijective function from A to $A \cup B$. (Why?)

It follows that $A \sim A \cup B$.

Lemma (XXIV).

Let A, B be sets.

Suppose A is infinite and B is countably infinite. Further suppose $A \cap B = \emptyset$.

Then $A \cup B \sim A$.

Proof of Lemma (XXIV).

Let A, B be sets. Suppose A is infinite and B is countably infinite. Further suppose $A \cap B = \emptyset$.

There is an injective function from $\mathbb{N}$ to A.

This injective function defines some infinite sequence $\{x_n\}_{n=0}^{\infty}$ in A with no repeated terms:

- $x_m \neq x_n$ whenever $m \neq n$.

Write $S = \{x_n \mid n \in \mathbb{N}\}$. ($S$ is the set of all terms of $\{x_n\}_{n=0}^{\infty}$.)

Since $B \sim \mathbb{N}$, there is a bijective function from $\mathbb{N}$ to B.

This bijective function defines some infinite sequence $\{y_n\}_{n=0}^{\infty}$ in B, exhausting B and with no repeated terms:

- $B = \{y_n \mid n \in \mathbb{N}\}$, and $y_m \neq y_n$ whenever $m \neq n$.

Proof of Theorem (XXIV). (*Cont'd.*)

Idea. How to proceed with the construction of a bijective function from A to $A \cup B$?

$$\begin{array}{c||c|cccccccccc||c} A & S & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & \cdots & x_{2N} & x_{2N+1} & \cdots & z \in A \backslash S \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \cdots & \downarrow \\ A \cup B & S \cup B & y_0 & x_0 & y_1 & x_1 & y_2 & x_2 & \cdots & y_N & x_N & \cdots & z \in A \backslash S \end{array}$$

Define [with the table above in mind]

$$G_1 = \{(x_{2n}, y_n) \mid n \in \mathbb{N}\} \cup \{(x_{2n+1}, x_n) \mid n \in \mathbb{N}\}, \qquad G_2 = \{(z, z) \mid z \in A \backslash S\}$$

Define the relation $g = (A, A \cup B, G)$ by $G = G_1 \cup G_2$.

g is a bijective function from A to $A \cup B$. (Why?)

It follows that $A \sim A \cup B$.

3. **Consequences of 'Hilbert's Hotel'.**

The conclusion in Theorem (XXII) still holds when the condition '$A \cap B = \emptyset$' is dropped.

Corollary (XXV).

Let C, D be sets. Suppose C is infinite and D is countable. Then $C \cup D \sim C$.

Proof.

... Note that $C \cup D = C \cup (D \backslash C)$ and $C \cap (D \backslash C) = \emptyset$.

C is infinite and $D \backslash C$ is countable. (Why?)

Then, by Theorem (XXII), $C \cup D = C \cup (D \backslash C) \sim C$.

Corollary (XXVI).

Let C, D be sets. Suppose $D \subset C$.

Also suppose that $C \backslash D$ is infinite and D is countable.

Then $C \backslash D \sim C$.

Proof.

... Note that $C = (C \backslash D) \cup D$ and $(C \backslash D) \cap D = \emptyset$.

$C \backslash D$ is infinite and D is countable.

Then, by Theorem (XXII), $C = (C \backslash D) \cup D \sim C \backslash D$.

Example of application of Corollary (XXVI).

$\mathbb{R}\backslash\mathbb{Q}\sim\mathbb{R}$. Also, $\mathbb{Q} < \mathbb{R}\backslash\mathbb{Q}$.

Justification:

$\{\sqrt[n+2]{2}\}_{n=0}^{\infty}$ is an infinite sequence with no repeated terms in $\mathbb{R}\backslash\mathbb{Q}$. (Why?)
Therefore $\mathbb{R}\backslash\mathbb{Q}$ is infinite.
By Corollary (XXVI), $\mathbb{R}\backslash\mathbb{Q}\sim\mathbb{R}$.
Since $\mathbb{Q} < \mathbb{R}$, we also have $\mathbb{Q} < \mathbb{R}\backslash\mathbb{Q}$.

Remark.

There are as many irrational numbers as real numbers. There are 'much more' irrational numbers than rational numbers.

4. **Countable union of countable sets.**

Recall the definition for the notion of generalized union:

- *Let M be a set and $\{S_n\}_{n=0}^{\infty}$ be an infinite sequence of subsets of M. The (generalized) union of $\{S_n\}_{n=0}^{\infty}$ is defined to be the set $\{x \in A : x \in S_n$ for some $n \in \mathbb{N}\}$. It is denoted by $\bigcup_{n=0}^{\infty} S_n$.*

Theorem (XXVII). (Countability of countable union of countable sets.)

Let A be a set, and $\{A_n\}_{n=0}^{\infty}$ be an infinite sequence of countable subsets of A.

$\bigcup_{n=0}^{\infty} A_n$ is countable.

Remark.

Hence 'the (generalized) union of countably many countable sets is countable'.

Theorem (XXVII). (Countability of countable union of countable sets.)

Let A be a set.

Suppose $\{A_n\}_{n=0}^{\infty}$ is an infinite sequence of countable subsets of A.

Then $\bigcup_{n=0}^{\infty} A_n$ is countable.

Idea of proof.

Write $B = \bigcup_{n=0}^{\infty} A_n$.

For each $n \in \mathbb{N}$, label the elements of A_n exhaustively with elements of $\mathbb{N}$, so that we have $A_n = \{x_{n0}, x_{n1}, x_{n2}, \cdots\}$.

Now obtain the possibly 'infinite array', which exhausts the elements of the set B:

$$\begin{array}{c|ccccc} A_0 & x_{00} & x_{01} & x_{02} & x_{03} & \cdots \\ A_1 & x_{10} & x_{11} & x_{12} & x_{13} & \cdots \\ A_2 & x_{20} & x_{21} & x_{22} & x_{23} & \cdots \\ A_3 & x_{30} & x_{31} & x_{32} & x_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

For each n, the 'row' of the 'x_{nj}'s may 'terminate' or not.

There may be 'repeated' entries amongst the 'x_{ij}'s'.

No matter what, the 'size' of this 'array' is at most that of $\mathbb{N}^2$ and hence that of $\mathbb{N}$.

Hence B is countable.

Theorem (XXVII). (Countability of countable union of countable sets.)

Let A be a set.

Suppose $\{A_n\}_{n=0}^{\infty}$ be an infinite sequence of countable subsets of A.

Then $\bigcup_{n=0}^{\infty} A_n$ is countable.

Corollary (XXVIII). (Sufficiency criteria for being countably infinite.)

Let A be a set, and $\{A_n\}_{n=0}^{\infty}$ be an infinite sequence of countable subsets of A.

(1) *Suppose $\bigcup_{n=0}^{\infty} A_n$ is infinite. Then $\bigcup_{n=0}^{\infty} A_n$ is countably infinite.*

(2) *Suppose there exists some $m \in \mathbb{N}$ such that A_m is countably infinite.*

Then $\bigcup_{n=0}^{\infty} A_n$ is countably infinite.

(3) *Suppose there exists some infinite sequence $\{x_n\}_{n=0}^{\infty}$ in A such that both of the statements below hold:*

(3a) *($x_n \in A_n$ for any $n \in \mathbb{N}$) and*

(3b) *(for any $k, m \in \mathbb{N}$, if $k \neq m$ then $x_k \neq x_m$).*

Then $\bigcup_{n=0}^{\infty} A_n$ is countably infinite.

5. **Examples of applications of Theorem (XXVII), Theorem (XXVIII).**

(1) Another argument for $\mathbb{Q}\sim\mathbb{N}$:

- Write $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$.

  For any $n \in \mathbb{N}^*$, define $Q_n = \left\{\frac{m}{n}\middle|\ m \in \mathbb{Z}\right\}$.

  We have $\mathbb{Q} = \bigcup_{n=1}^{\infty} Q_n$. (Why?)

  Note that $Q_n\sim\mathbb{Z}\sim\mathbb{N}$. (Why?)

  Then $\mathbb{Q}\lesssim\mathbb{N}$.

  Recall $\mathbb{N}\lesssim\mathbb{Q}$.

  Then we have $\mathbb{Q}\sim\mathbb{N}$.

(2) Denote by $\mathbb{Q}[x]$ the set of all polynomials with indeterminate x and with coefficients in $\mathbb{Q}$.

$\mathbb{Q}[x]\backslash\{0\}\sim\mathbb{N}$. Also, $\mathbb{Q}[x]\sim\mathbb{N}$.

Justification:

- For any $n \in \mathbb{N}$, define

$$T_n = \{f(x) \in \mathbb{Q}[x] : \deg(f(x)) = n\}.$$

(${T_n}$ is the set of all degree-n polynomials with indeterminate x and with coefficients in $\mathbb{Q}$.)

We have $\mathbb{Q}[x]\backslash\{0\} = \bigcup_{n=0}^{\infty} T_n$. (Why?)

Write $\mathbb{Q}^* = \mathbb{Q}\backslash\{0\}$. Note that $T_n\sim\mathbb{Q}^* \times \mathbb{Q}^n\sim\mathbb{N}$. (How? Why?)

Then $\mathbb{Q}[x]\backslash\{0\}\lesssim\mathbb{N}$.

For each $j \in \mathbb{N}$, denote by x^j the monic monomial of degree j.

$\mathbb{N}\sim\{x^j \mid j \in \mathbb{N}\}\lesssim\mathbb{Q}[x]\backslash\{0\}$.

Then we have $\mathbb{Q}[x]\backslash\{0\}\sim\mathbb{N}$.

Then $\mathbb{Q}[x]\sim\mathbb{N}$ also. (Why?)

(3) Define $\mathbb{A} = \{\zeta \in \mathbb{C} : \zeta \text{ is a root of } f(x) \text{ for some } f(x) \in \mathbb{Q}[x]\backslash\{0\}\}$.

The elements of $\mathbb{A}$ are called **algebraic numbers (over $\mathbb{Q}$)**.

The elements of $\mathbb{C}\backslash\mathbb{A}$ are called **transcendental numbers (over $\mathbb{Q}$)**.

$\mathbb{A} \sim \mathbb{N}$.

Justification:

- For any $f(x) \in \mathbb{Q}[x]\backslash\{0\}$, define $Z[f(x)] = \{\zeta \in \mathbb{C} : \zeta \text{ is a root of } f(x)\}$.
  ($Z[f(x)]$ is the set of all roots of $f(x)$ in $\mathbb{C}$.)
  $Z[f(x)]$ is finite, and hence $Z[f(x)] \lesssim \mathbb{N}$.
  Since $\mathbb{Q}[x]\backslash\{0\}$ is countably infinite, we may 'arrange' all the elements of $\mathbb{Q}[x]\backslash\{0\}$ as an infinite sequence without repeating terms $\{f_n(x)\}_{n=0}^{\infty}$, so that

$$\mathbb{Q}[x]\backslash\{0\} = \{f_n(x) \mid n \in \mathbb{N}\}.$$

  Now $\mathbb{A} = \bigcup_{n=0}^{\infty} Z[f_n(x)]$. Then $A \lesssim \mathbb{N}$.

  Note that $\mathbb{N} \lesssim \mathbb{A}$. Therefore we have $\mathbb{A} \sim \mathbb{N}$.

Remark. It follows that $\mathbb{A} < \mathbb{C}$ and $\mathbb{A} \cap \mathbb{R} < \mathbb{R}$.

There are real/complex numbers which are not algebraic; actually there are much more (real/complex) transcendental numbers than there are (real/complex) algebraic numbers.

This is a non-constructive proof of the existence of transcendenal numbers.

6. $\mathbb{N}$ is the ‘smallest’ infinite set.

By Cantor’s Theorem, we have $\mathbb{N} < \mathfrak{P}(\mathbb{N}) < \mathfrak{P}(\mathfrak{P}(\mathbb{N})) < \cdots$.

Two further questions on the ‘chain’ $\mathbb{N} < \mathfrak{P}(\mathbb{N}) < \mathfrak{P}(\mathfrak{P}(\mathbb{N})) < \cdots$.

Question (1).

Is there a set of cardinality greater than each of $\mathbb{N}$, $\mathfrak{P}(\mathbb{N})$, $\mathfrak{P}(\mathfrak{P}(\mathbb{N}))$, ...*?*

Answer. Yes, one such set is the ‘union’ of all these sets.

To make sense of this set, we need the Axiom of Substitution.

$\mathbb{N}$ is the 'smallest' infinite set.

By Cantor's Theorem, we have $\mathbb{N} < \mathfrak{P}(\mathbb{N}) < \mathfrak{P}(\mathfrak{P}(\mathbb{N})) < \cdots$.

Two further questions on the 'chain' $\mathbb{N} < \mathfrak{P}(\mathbb{N}) < \mathfrak{P}(\mathfrak{P}(\mathbb{N})) < \cdots$.

Question (1).

Is there a set of cardinality greater than each of $\mathbb{N}$, $\mathfrak{P}(\mathbb{N})$, $\mathfrak{P}(\mathfrak{P}(\mathbb{N}))$, ...*?*

Question (2).

Is there a set of cardinality greater than $\mathbb{N}$ *and less than* $\mathbb{R}$*?*

Answer. Cantor believed there was no such set.

Cantor's Continuum Hypothesis:

For any set S, *if* $\mathbb{N} \lesssim S \lesssim \mathbb{R}$ *then* $(S \sim \mathbb{N}$ *or* $S \sim \mathbb{R})$.

So what are $\mathbb{N}$ and $\mathbb{R}$, really?

Or, what is the respective nature of these two sets?

This leads us to the foundation of mathematics.