MATH1050 Finite sets versus infinite sets

1. **Definition.**

   Let A be a set. A is said to be **finite** *if there exists some $n_A \in \mathbb{N}$ such that $A \sim [\![1, n_A]\!]$. The number n_A is called the cardinality of A.*

   Remark. The number n_A is uniquely determined by A. (Proof?)

2. **Illustration of properties of finite sets through an example**.

   Consider the set $\{1, 2, 3\}$.

   (1) *Question.*

   Is there some injective function from $\mathbb{N}$ to $\{1, 2, 3\}$? Is there an infinite sequence with no repeated terms in $\{1, 2, 3\}$?

   Answer and reason.

   No. Were there an infinite sequence $(a_n)_{n=0}^{\infty}$ with no repeated terms in $\{1, 2, 3\}$, we would have $\{a_0, a_1, a_2\} = \{1, 2, 3\}$ and then a_3 would have to be one of a_0, a_1, a_2.

   (2) *Question.*

   Is there some proper subset U of $\{1, 2, 3\}$ satisfying $\{1, 2, 3\} \sim U$?

   Answer and reason.

   No. (Heuristic argument.) Every subset of $\{1, 2, 3\}$ has at most two elements, but $\{1, 2, 3\}$ has three elements. They are not 'of the same size'.

   (3) *Question.*

   Is there some function $\varphi : \{1, 2, 3\} \longrightarrow \{1, 2, 3\}$ which is injective and not surjective?

   Answer and reason.

   No. There are six injective functions from $\{1, 2, 3\}$ to $\{1, 2, 3\}$. They are given by:

   $$\begin{array}{ll} \varphi_1(1) = 1,\ \varphi_1(2) = 2,\ \varphi_1(3) = 3. & \varphi_4(1) = 1,\ \varphi_4(2) = 3,\ \varphi_4(3) = 2. \\ \varphi_2(1) = 2,\ \varphi_2(2) = 3,\ \varphi_2(3) = 1. & \varphi_5(1) = 2,\ \varphi_5(2) = 1,\ \varphi_5(3) = 3. \\ \varphi_3(1) = 3,\ \varphi_3(2) = 1,\ \varphi_3(3) = 2. & \varphi_6(1) = 3,\ \varphi_6(2) = 2,\ \varphi_6(3) = 1. \end{array}$$

   They are all surjective.

   (4) *Question.*

   Is there some function $\psi : \{1, 2, 3\} \longrightarrow \{1, 2, 3\}$ which is surjective and not injective?

   Answer and reason.

   No. There are six surjective functions from $\{1, 2, 3\}$ to $\{1, 2, 3\}$. They are given by:

   $$\begin{array}{ll} \psi_1(1) = 1,\ \psi_1(2) = 2,\ \psi_1(3) = 3. & \psi_4(1) = 1,\ \psi_4(2) = 3,\ \psi_4(3) = 2. \\ \psi_2(1) = 2,\ \psi_2(2) = 3,\ \psi_2(3) = 1. & \psi_5(1) = 2,\ \psi_5(2) = 1,\ \psi_5(3) = 3. \\ \psi_3(1) = 3,\ \psi_3(2) = 1,\ \psi_3(3) = 2. & \psi_6(1) = 3,\ \psi_6(2) = 2,\ \psi_6(3) = 1. \end{array}$$

   They are all injective.

   We can ask the same questions for every other finite set, and obtain the same answers along the same line of reasoning.

3. **Theorem (XIX). (Characterization of finite sets.)**

   Let A be a set. The statements below are equivalent:

   (1) *A is finite.*

   (2) *No proper subset of A is of cardinality equal to A.*

   (3) *For any function φ from A to A, if φ is injective then φ is surjective.*

   (4) *For any function ψ from A to A, if ψ is surjective then ψ is injective.*

   Proof of Theorem (XIX). A very tedious exercise in mathematical induction.

4. **Definition.**

   Let A be a set. A is said to be **infinite** *if $\mathbb{N} \lesssim A$.*

   Remark. Heuristic idea in this definition: A is infinite iff A contains at least a 'copy' of $\mathbb{N}$ as a subset.

   Examples. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, $[0, 1]$, line segments, circles, squares, discs in the plane $\mathbb{R}^2$, ...

5. **Illustration of properties of infinite sets through an example**.

   Consider the set $[0, 1]$.

   (1) *Question.*
   Is there some injective function from $\mathbb{N}$ to $[0, 1]$? Is there an infinite sequence with no repeated terms in $[0, 1]$?
   Answer and reason.
   Yes. One such infinite sequence is $\{1/2^n\}_{n=0}^{\infty}$. So an injective function g from $\mathbb{N}$ to $[0, 1]$ is given by $g(n) = 1/2^n$ for any $n \in \mathbb{N}$.

   (2) *Question.*
   Is there some proper subset U of $[0, 1]$ satisfying $[0, 1]\sim U$?
   Answer and reason.
   Yes. One such set is $[0, 1)$.

   (3) *Question.*
   Is there some function $\varphi : [0, 1] \longrightarrow [0, 1]$ which is injective and not surjective?
   Answer and reason.
   Yes. One such function is given by $\varphi : [0, 1] \longrightarrow [0, 1]$, where $S = \{1/2^n \mid n \in \mathbb{N}\}$ and

   $$\varphi(x) = \begin{cases} x & \text{if} \quad x \in [0, 1]\backslash S \\ x/2 & \text{if} \quad x \in S \end{cases}$$

   (Note that for any $x \in [0, 1]$, we have $\varphi(x) \neq 1$.)

   (4) *Question.*
   Is there some function $\psi : [0, 1] \longrightarrow [0, 1]$ which is surjective and not injective?
   Answer and reason.
   Yes. One such function is given by $\psi : [0, 1] \longrightarrow [0, 1]$, where $S = \{1/2^n \mid n \in \mathbb{N}\}$ and

   $$\psi(x) = \begin{cases} x & \text{if} \quad x \in [0, 1]\backslash S \\ 2x & \text{if} \quad x \in S\backslash\{1\} \\ 1 & \text{if} \quad x = 1 \end{cases}$$

   (Note that we have $\psi(1) = \psi(1/2) = 1$.)

   We can ask the same questions for every other infinite set, and and obtain the same answers along the same line of reasoning.

6. **Theorem (XX). (Characterization of infinite sets.)**

   Let A be a set. The statements below are equivalent:

   (1) *A is infinite. ($\mathbb{N} \lesssim A$.)*

   (1') *There exists some subset S of A such that $\mathbb{N}\sim S$.*

   (1") *There exists some subset T of A such that $\mathbb{N} \lesssim T$.*

   (2) *There exists some proper subset U of A such that $A\sim U$.*

   (2') *There exists some proper subset V of A such that $A \lesssim V$.*

   (3) *There exists some function φ from A to A such that φ is injective and φ is not surjective.*

   (4) *There exists some function ψ from A to A such that ψ is surjective and ψ is not injective.*

   Remark. By Theorem (XIX) and Theorem (XX), every set is finite or infinite, but not both.

7. **Proof of Theorem (XX).**

Let A be a set.

(a) Suppose A is infinite.

[We want to deduce that there exists some proper subset U of A such that $A \sim U$.]

Pick some injective function $g : \mathbb{N} \longrightarrow A$.

Define $z = g(0)$. Define $U = A \backslash \{z\}$.

Note that U is a proper subset of A.

Define the relation $h = (A, U, H)$ by

$$H = \{(x, x) \mid x \in A \backslash g(\mathbb{N})\} \cup \{(g(n), g(n+1)) \mid n \in \mathbb{N}\}$$

h is a bijective function from A to U. Hence $A \sim U$.

[Note that the function $\mathsf{id}_{A \backslash g(\mathbb{N})}$ is bijectgive. Also note that the function from $g(\mathbb{N})$ to $g(\mathbb{N}) \backslash \{z\}$ given by $n \longmapsto n+1$ for any $n \in \mathbb{N}$ is bijective. The bijective function h is obtained by glueing these two bijective functions according the Glueing Lemma.]

(b) Suppose there exists some proper subset U of A such that $A \sim U$.

[We want to deduce that there exists some function φ from A to A such that φ is injective and φ is not surjective.]

Pick some bijective function $h : A \longrightarrow U$.

Denote by $\iota_U : U \longrightarrow A$ by $\iota_U(x) = x$ for any $x \in U$.

Define the function $\varphi : A \longrightarrow A$ by $\varphi = \iota_U \circ h$.

Since h, ι_U are both injective, φ is also injective.

Note that $\varphi(A) = (\iota_U \circ h)(A) = \iota_U(h(A)) = \iota_U(U) = U \subsetneq A$. Then φ is not surjective.

(c) Suppose there exists some function φ from A to A such that φ is injective and φ is not surjective.

[We want to deduce that A is infinite according to definition. We are going to contruct an infinite sequence in A with no repeated terms.]

Since φ is not surjective, $A \backslash \varphi(A) \neq \emptyset$.

Pick some $b \in A \backslash \varphi(A)$. Define $b_0 = b$. For any $m \in \mathbb{N} \backslash \{0\}$, we define $b_m = \varphi(b_{m-1})$.

By definition, for each $n \in \mathbb{N}$ we have $b_n = \underbrace{(\varphi \circ \varphi \circ \cdots \circ \varphi \circ \varphi)}_{n \text{ times}}(b)$. (Why? Apply mathematical induction.)

Then, by the injectivity of φ, we have $b_j \neq b_k$ whenever $j \neq k$. (Why?)

It follows that the function $f : \mathbb{N} \longrightarrow A$ defined by $f(n) = b_n$ for any $n \in \mathbb{N}$ is an injective function.

Therefore $\mathbb{N} \lesssim A$. Hence A is infinite.

(d) Suppose there exists some function φ from A to A such that φ is injective and φ is not surjective.

[We want to deduce that there exists some function ψ from A to A such that ψ is surjective and ψ is not injective.]

Note that $A \neq \emptyset$. (Why?)

Define the function $\hat{\varphi} : A \longrightarrow \varphi(A)$ by $\hat{\varphi}(x) = \varphi(x)$ for any $x \in A$.

By the injectivity of φ, the function $\hat{\varphi}$ is injective.

By the definition of $\hat{\varphi}$, we have $\hat{\varphi}(A) = \varphi(A)$. Then $\hat{\varphi}$ is surjective.

Therefore $\hat{\varphi}$ is a bijective function.

Since φ is not surjective, $A \backslash \varphi(A) \neq \emptyset$. Pick some $z_0 \in A \backslash \varphi(A)$.

Define the function $\psi : A \longrightarrow A$ by

$$\psi(y) = \begin{cases} \hat{\varphi}^{-1}(y) & \text{if} \quad y \in \varphi(A) \\ z_0 & \text{if} \quad y \in A \backslash \varphi(A) \end{cases}$$

We have $\psi(A) \supset \psi(\varphi(A)) = \psi(\hat{\varphi}(A)) = \hat{\varphi}^{-1}(\hat{\varphi}(A)) = A$.

Then ψ is surjective.

We verify that ψ is not injective:

- Since $z_0 \in A \backslash \varphi(A)$, we have $z_0 \neq \varphi(z_0)$.
  We have $\psi(z_0) = z_0$ and $\psi(\varphi(z_0)) = \hat{\varphi}^{-1}(\hat{\varphi}(z_0)) = z_0$.
  Then $\psi(z_0) = \psi(\varphi(z_0))$.
  Therefore ψ is not injective.

(e) Suppose there exists some function ψ from A to A such that ψ is surjective and ψ is not injective.

[We want to deduce that there exists some function φ from A to A such that φ is injective and φ is not surjective.]

By the surjectivity of ψ, for any $x \in A$, there exists some $y_x \in A$ such that $\psi(y_x) = x$. [Here we need the Axiom of Choie.]

Define the function $\varphi : A \longrightarrow A$ by $\varphi(x) = y_x$ for any $x \in A$.

We verify that φ is injective:

- Pick any $x, x' \in A$. Suppose $\varphi(x) = \varphi(x')$. Then $y_x = y_{x'}$. Therefore $x = \varphi(y_x) = \varphi(y_{x'}) = x'$.
  So φ is injective.

We verify that φ is not surjective:

- Since ψ is not injective, there exist some $x_0, z_0 \in A$ such that $y_{x_0} \neq z_0$ and $\psi(y_{x_0}) = \psi(z_0)$.
  By the definition of z_0, we have $\psi(z_0) = x_0$ and $z_0 \neq y_{x_0}$. Then by the definition of φ, we have $z_0 \notin \varphi(A)$.
  So φ is not surjective.