1. **Definition.**

Let A be a set.

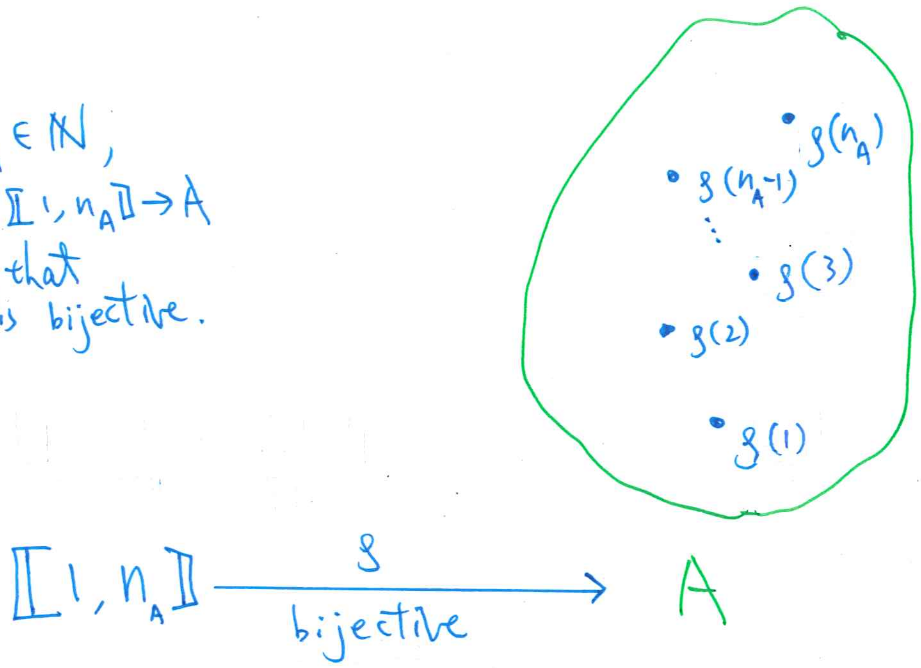
A is said to be **finite** *if there exists some $n_A \in \mathbb{N}$ such that $A \sim [\![1, n_A]\!]$.*

The number n_A is called the cardinality of A.

Remark.

The number n_A is uniquely determined by A. (Proof?)

$\exists n_A \in \mathbb{N}$,
$g : [\![1, n_A]\!] \to A$
such that
g is bijective.



$A = \{g(1), g(2), g(3), \dots, g(n_A)\}$

For any $k, m \in [\![1, n_A]\!]$,
if $k \neq m$ then $g(k) \neq g(m)$.

So A has exactly n_A elements.

Convention: $[\![1, 0]\!] = \emptyset$.

Hence $n_A = 0$ iff $A = \emptyset$.

2. **Illustration of properties of finite sets through an example.**

Consider the set $\{1, 2, 3\}$.

(1) *Question.*

Is there some injective function from $\mathbb{N}$ to $\{1, 2, 3\}$? Is there an infinite sequence with no repeated terms in $\{1, 2, 3\}$?

Answer and reason.

No. Were there an infinite sequence $(a_n)_{n=0}^{\infty}$ with no repeated terms in $\{1, 2, 3\}$, we would have $\{a_0, a_1, a_2\} = \{1, 2, 3\}$ and then a_3 would have to be one of a_0, a_1, a_2.

(2) *Question.*

Is there some proper subset U of $\{1, 2, 3\}$ satisfying $\{1, 2, 3\} \sim U$?

Answer and reason.

No. (Heuristic argument.) Every subset of $\{1, 2, 3\}$ has at most two elements, but $\{1, 2, 3\}$ has three elements. They are not 'of the same size'.

(3) *Question.*

Is there some function $\varphi : \{1,2,3\} \longrightarrow \{1,2,3\}$ which is injective and not surjective?

Answer and reason.

No. There are six injective functions from $\{1,2,3\}$ to $\{1,2,3\}$. They are given by:

$$\varphi_1(1) = 1,\ \varphi_1(2) = 2,\ \varphi_1(3) = 3. \qquad \varphi_4(1) = 1,\ \varphi_4(2) = 3,\ \varphi_4(3) = 2.$$
$$\varphi_2(1) = 2,\ \varphi_2(2) = 3,\ \varphi_2(3) = 1. \qquad \varphi_5(1) = 2,\ \varphi_5(2) = 1,\ \varphi_5(3) = 3.$$
$$\varphi_3(1) = 3,\ \varphi_3(2) = 1,\ \varphi_3(3) = 2. \qquad \varphi_6(1) = 3,\ \varphi_6(2) = 2,\ \varphi_6(3) = 1.$$

They are all surjective.

(4) *Question.*

Is there some function $\psi : \{1,2,3\} \longrightarrow \{1,2,3\}$ which is surjective and not injective?

Answer and reason.

No. There are six surjective functions from $\{1,2,3\}$ to $\{1,2,3\}$. They are given by:

$$\psi_1(1) = 1,\ \psi_1(2) = 2,\ \psi_1(3) = 3. \qquad \psi_4(1) = 1,\ \psi_4(2) = 3,\ \psi_4(3) = 2.$$
$$\psi_2(1) = 2,\ \psi_2(2) = 3,\ \psi_2(3) = 1. \qquad \psi_5(1) = 2,\ \psi_5(2) = 1,\ \psi_5(3) = 3.$$
$$\psi_3(1) = 3,\ \psi_3(2) = 1,\ \psi_3(3) = 2. \qquad \psi_6(1) = 3,\ \psi_6(2) = 2,\ \psi_6(3) = 1.$$

They are all injective.

We can ask the same questions for every other finite set, and obtain the same answers along the same line of reasoning.

3. **Theorem (XIX). (Characterization of finite sets.)**

Let A be a set. The statements below are equivalent:

(1) *A is finite.*

(2) *No proper subset of A is of cardinality equal to A.*

(3) *For any function φ from A to A, if φ is injective then φ is surjective.*

(4) *For any function ψ from A to A, if ψ is surjective then ψ is injective.*

Proof of Theorem (XIX). A very tedious exercise in mathematical induction.

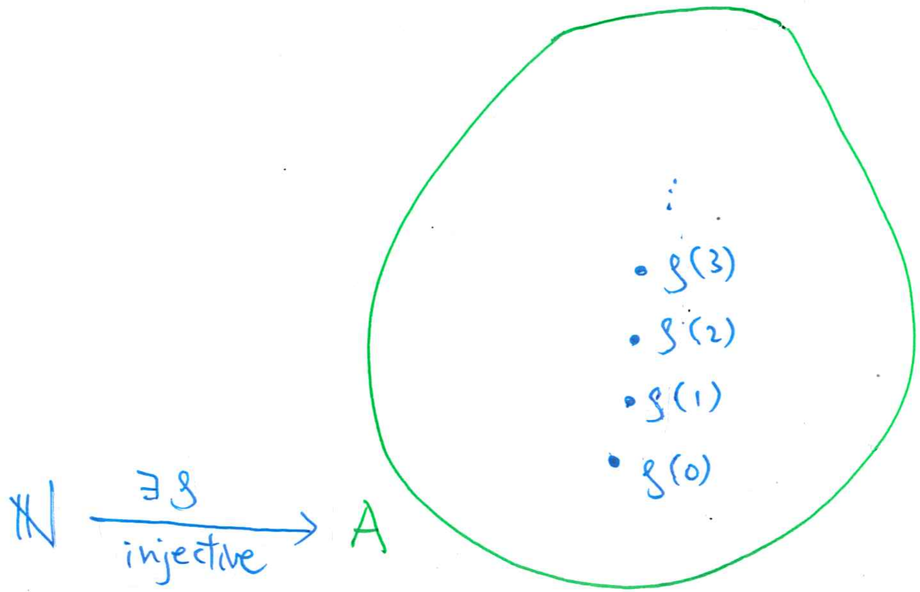
4. **Definition.**

Let A be a set.

A is said to be **infinite** *if $\mathbb{N} \lesssim A$.*

Remark.

Heuristic idea in this definition: A is infinite iff A contains at least a 'copy' of $\mathbb{N}$ as a subset.



$g(\mathbb{N}) = \{g(0), g(1), g(2), g(3), \ldots\}$
is a 'copy' of $\mathbb{N}$
in the sense that
$x \longmapsto g(x)$ for any $x \in \mathbb{N}$
defines a bijective function
from $\mathbb{N}$ to $g(\mathbb{N})$.

Examples.

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $[0, 1]$, line segments, circles, squares, discs in the plane $\mathbb{R}^2$, ...

5. **Illustration of properties of infinite sets through an example.**

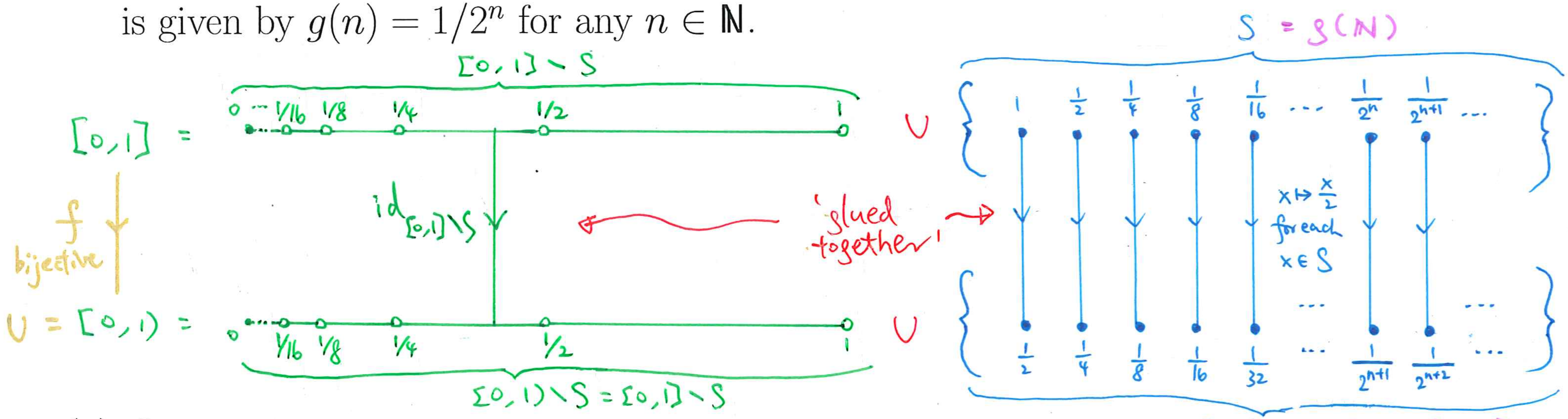
Consider the set $[0,1]$.

(1) *Question.*

Is there some injective function from $\mathbb{N}$ to $[0,1]$? Is there an infinite sequence with no repeated terms in $[0,1]$?

Answer and reason.

Yes. One such infinite sequence is $\{1/2^n\}_{n=0}^{\infty}$. So an injective function g from $\mathbb{N}$ to $[0,1]$ is given by $g(n) = 1/2^n$ for any $n \in \mathbb{N}$.



(2) *Question.*

Is there some proper subset U of $[0,1]$ satisfying $[0,1] \sim U$?

Answer and reason.

Yes. One such set is $[0,1)$.

(3) *Question.*

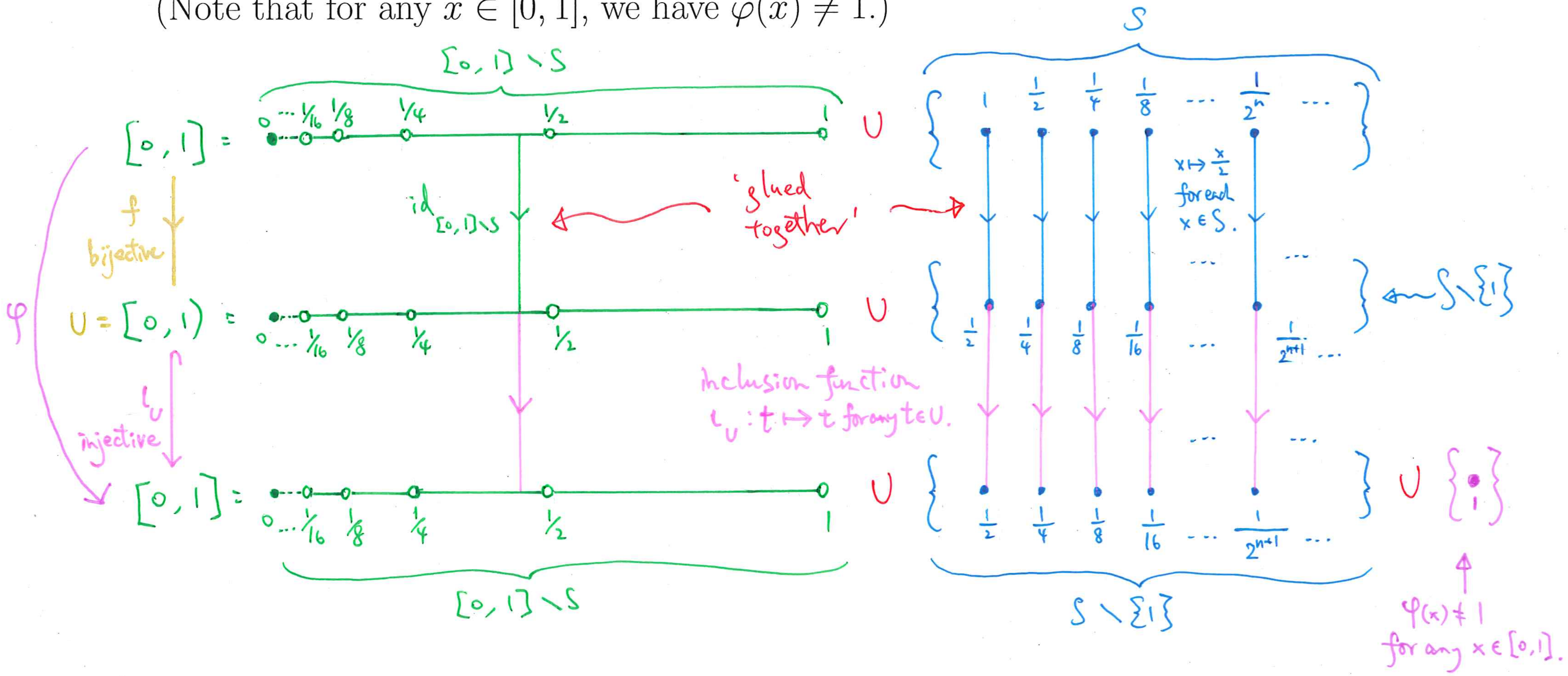
Is there some function $\varphi : [0,1] \longrightarrow [0,1]$ which is injective and not surjective?

Answer and reason.

Yes. One such function is given by $\varphi : [0,1] \longrightarrow [0,1]$, where $S = \{1/2^n \mid n \in \mathbb{N}\}$ and

$$\varphi(x) = \begin{cases} x & \text{if } x \in [0,1] \setminus S \\ x/2 & \text{if } x \in S \end{cases}.$$

(Note that for any $x \in [0,1]$, we have $\varphi(x) \neq 1$.)



(4) *Question.*

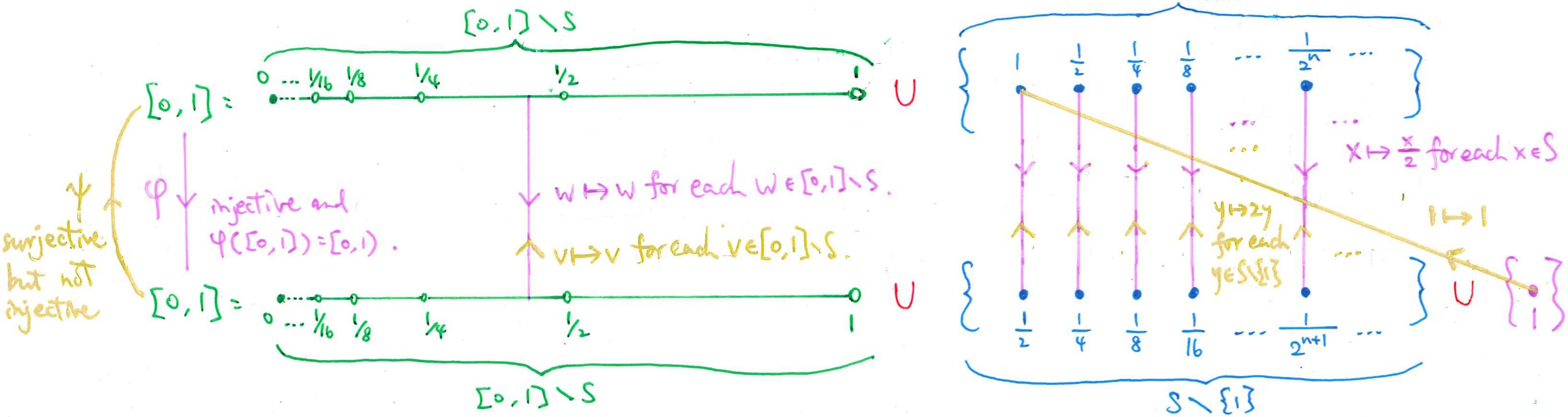
Is there some function $\psi : [0,1] \longrightarrow [0,1]$ which is surjective and not injective?

Answer and reason.

Yes. One such function is given by $\psi : [0,1] \longrightarrow [0,1]$, where $S = \{1/2^n \mid n \in \mathbb{N}\}$ and

$$\psi(x) = \begin{cases} x & \text{if } x \in [0,1]\backslash S \\ 2x & \text{if } x \in S\backslash\{1\} \\ 1 & \text{if } x = 1 \end{cases} .$$

(Note that we have $\psi(1) = \psi(1/2) = 1$.)



We can ask the same questions for every other infinite set, and and obtain the same answers along the same line of reasoning.

6. **Theorem (XX). (Characterization of infinite sets.)**

Let A be a set. The statements below are equivalent:

(1) *A is infinite. ($\mathbb{N} \lesssim A$.)*

(1') *There exists some subset S of A such that $\mathbb{N} \sim S$.*

(1") *There exists some subset T of A such that $\mathbb{N} \lesssim T$.*

(2) *There exists some proper subset U of A such that $A \sim U$.*

(2') *There exists some proper subset V of A such that $A \lesssim V$.*

(3) *There exists some function φ from A to A such that φ is injective and φ is not surjective.*

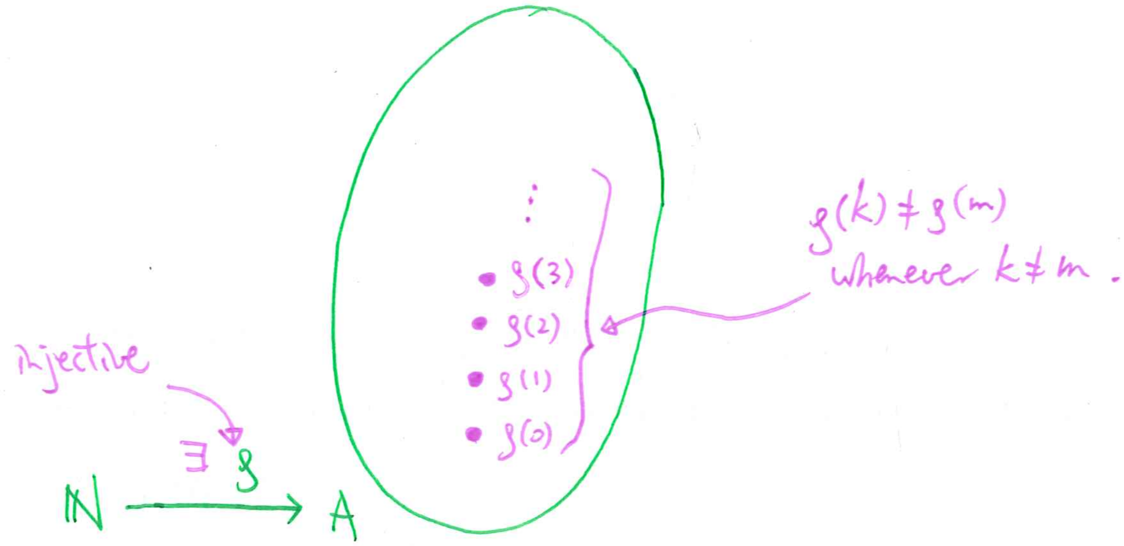
(4) *There exists some function ψ from A to A such that ψ is surjective and ψ is not injective.*

Remark.

By Theorem (XIX) and Theorem (XX), every set is finite or infinite, but not both.

Assumption:

A is infinite.



Conclusion.

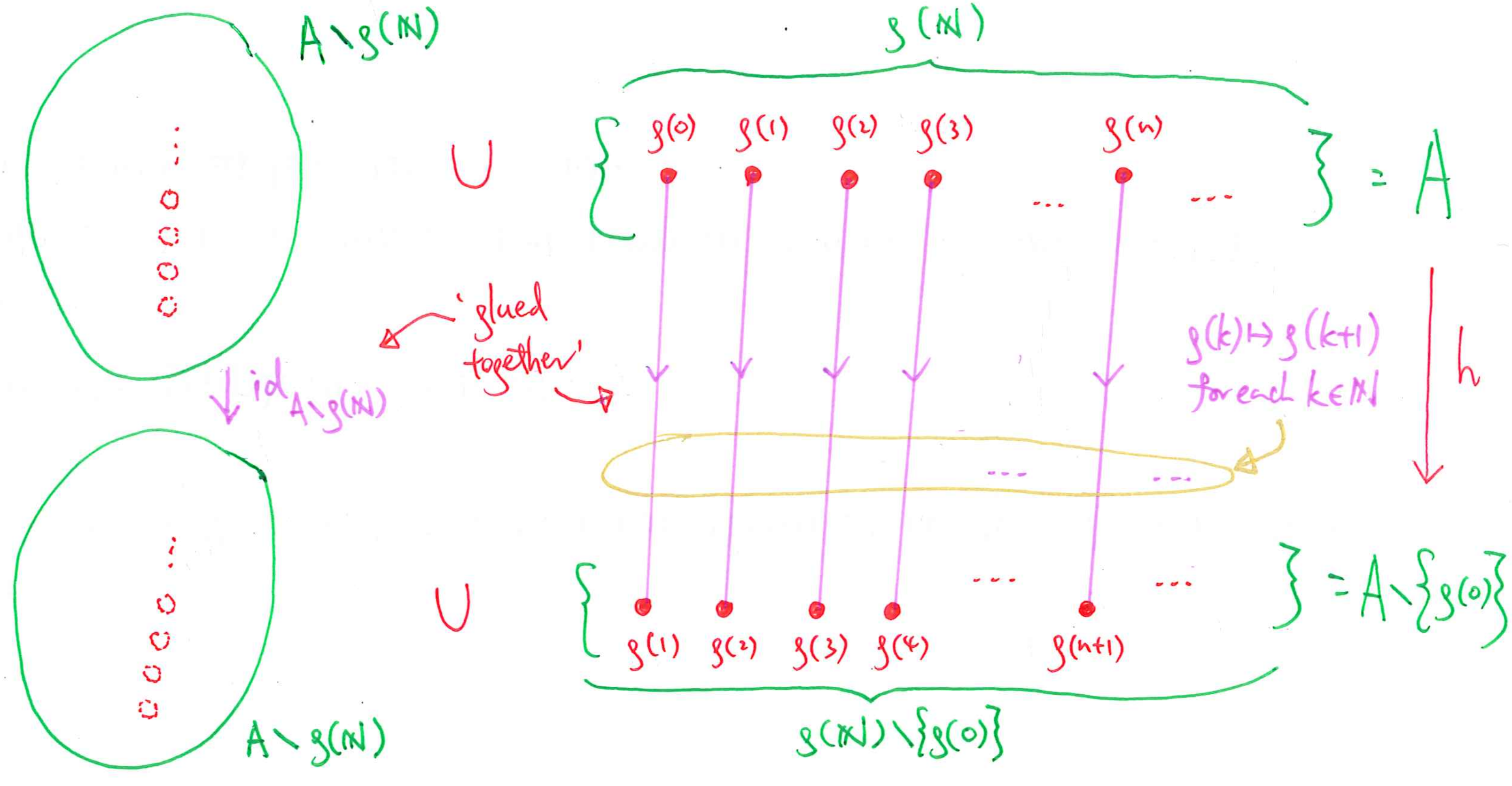
The function

$h: A \to A \setminus \{g(0)\}$

defined by

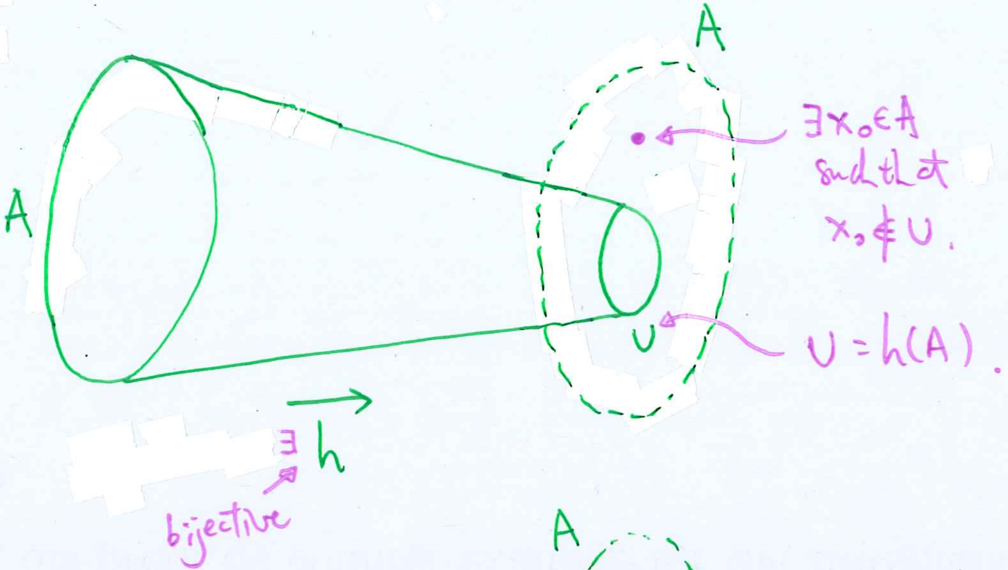
$$h(x) = \begin{cases} x & \text{if } x \notin g(\mathbb{N}) \\ g(n+1) & \text{if } x = g(n) \text{ for some } n \in \mathbb{N} \end{cases}$$

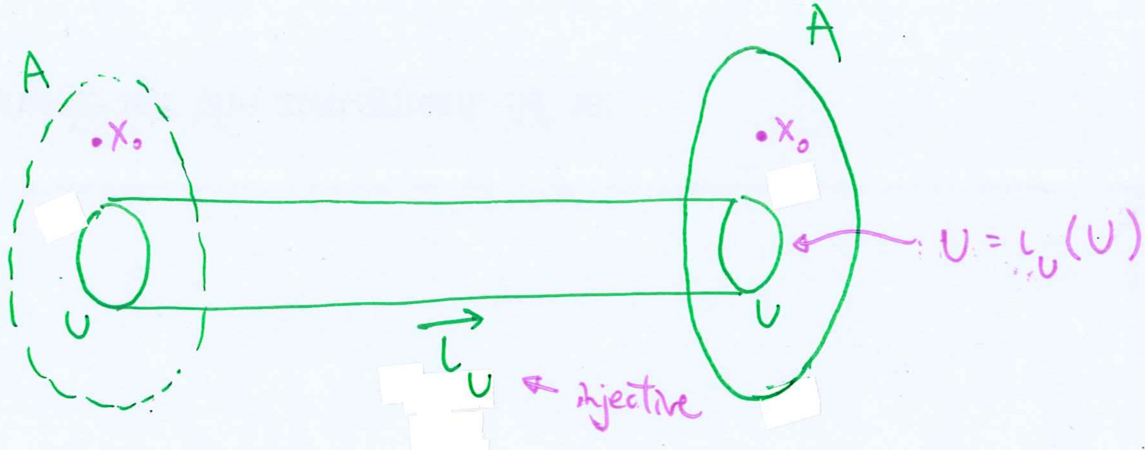
is a bijective function.



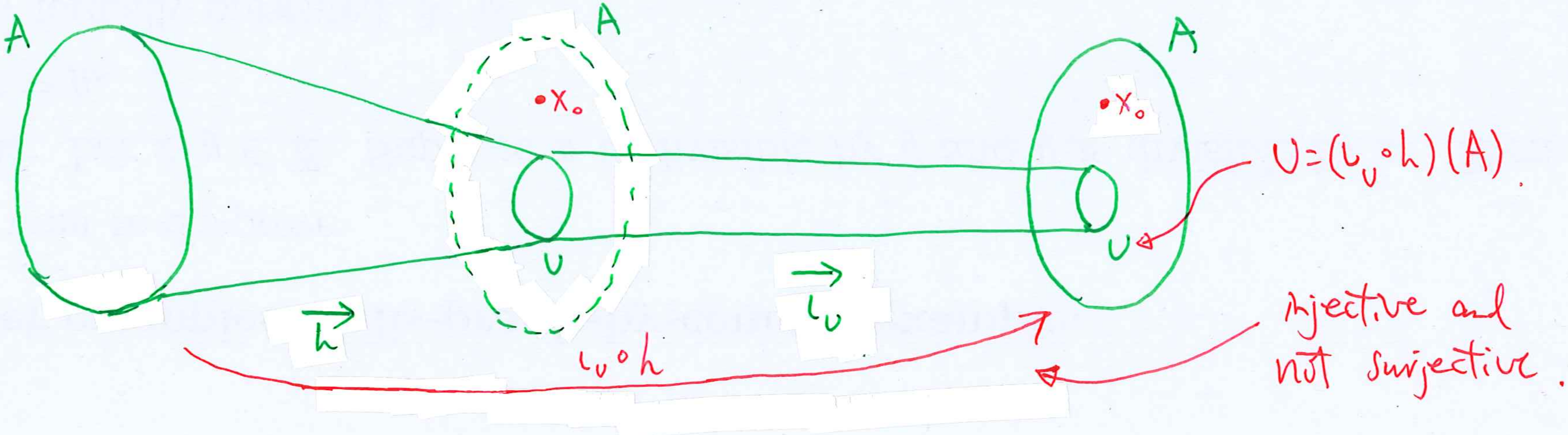
Assumption.

There exists some subset U of A such that $\begin{cases} A \sim U \\ U \subsetneq A. \end{cases}$





Conclusion.

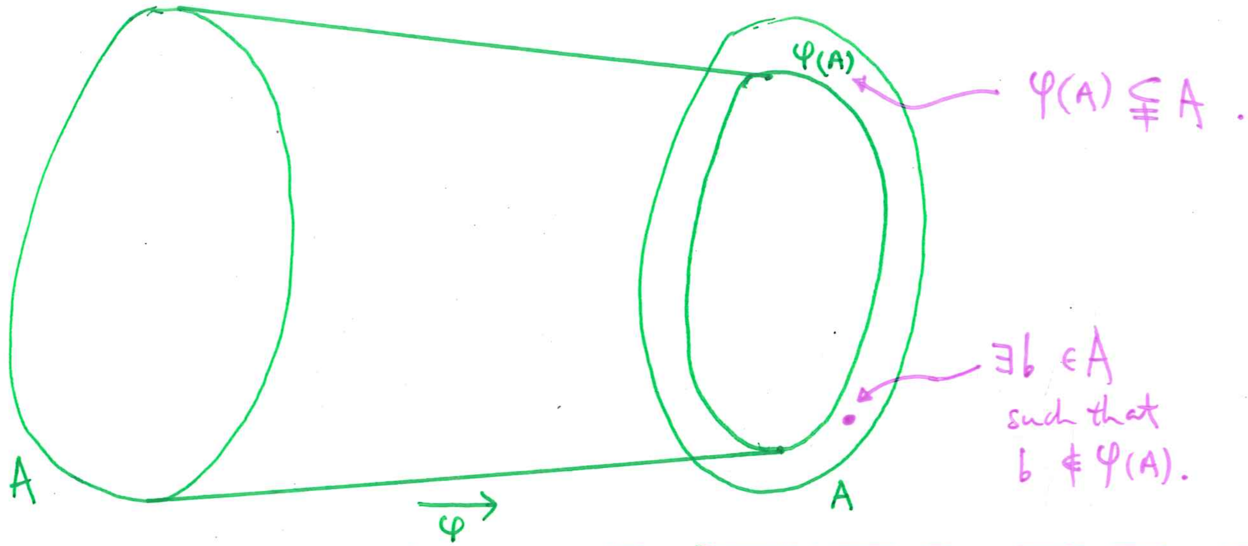


Assumption.

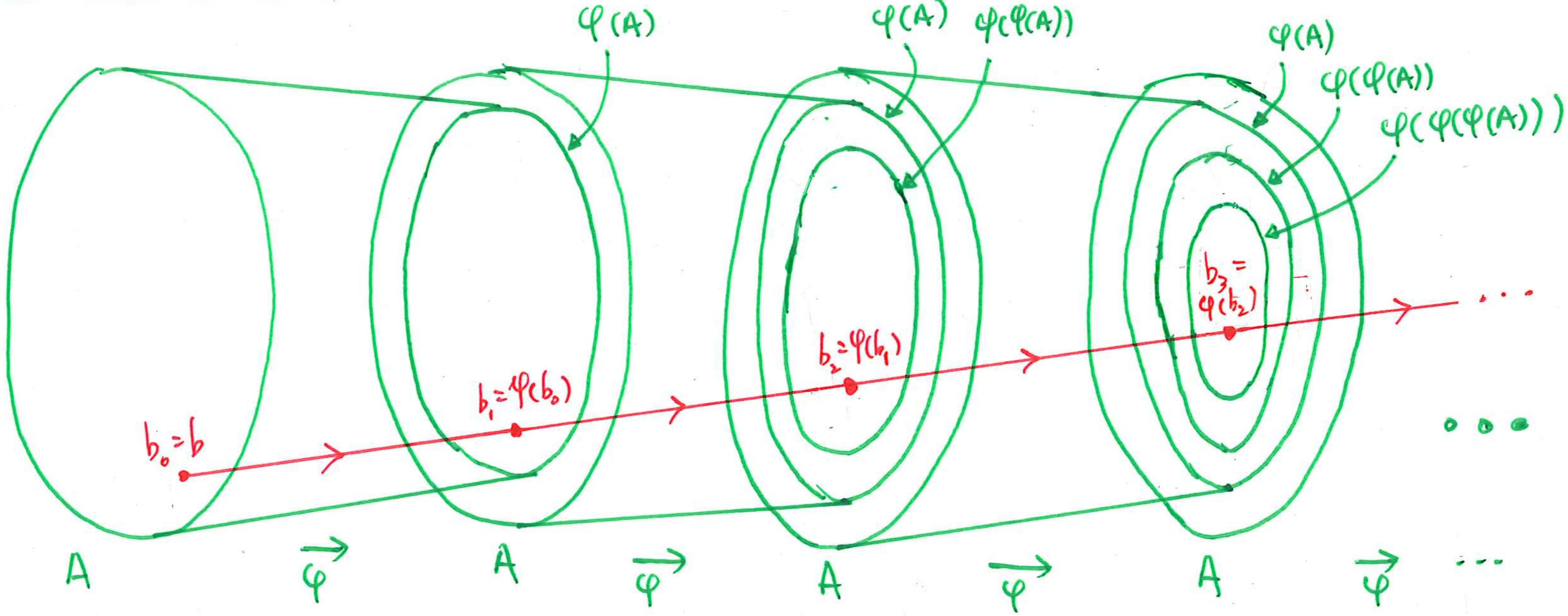
There exists some function $\varphi : A \to A$ such that

φ is injective and

φ is not surjective.



Conclusion.



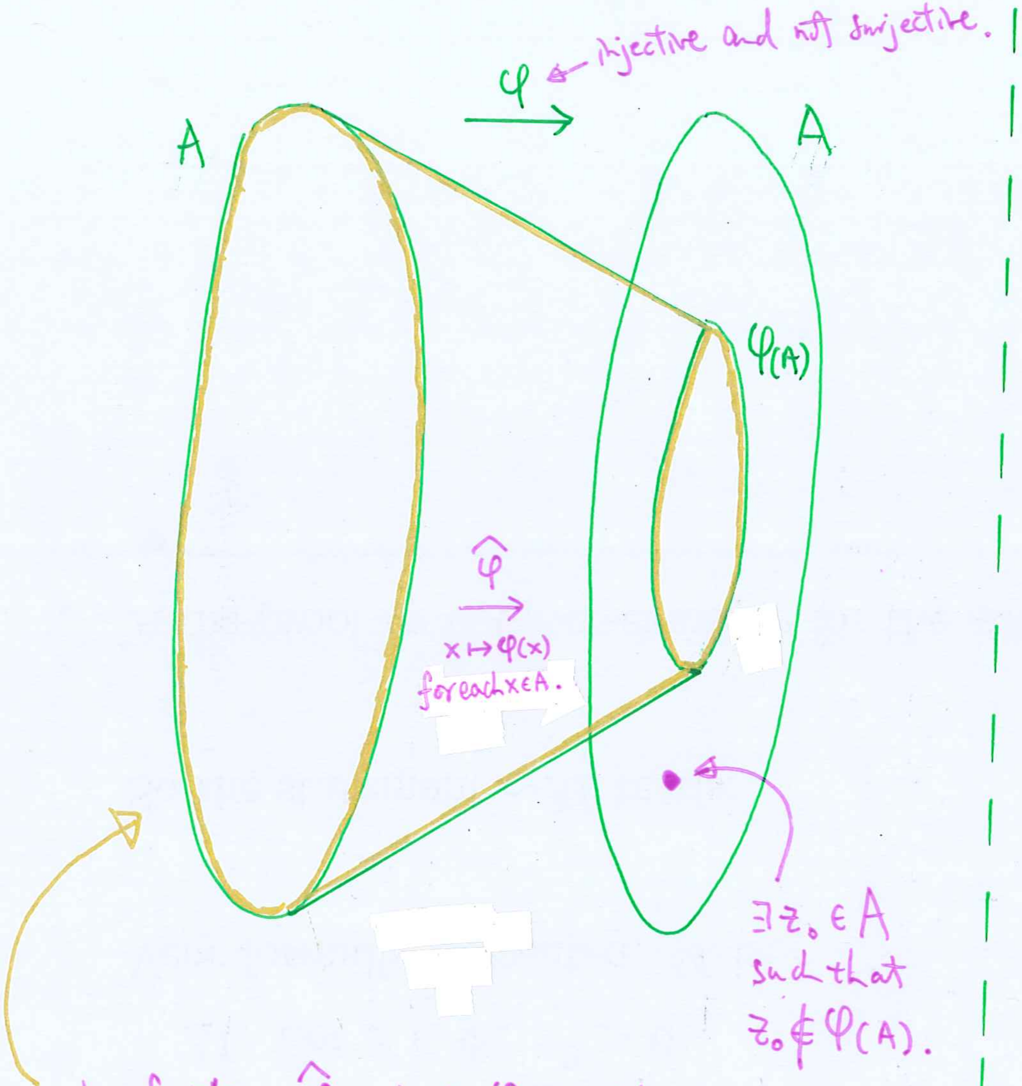
The infinite sequence $\{b_n\}_{n=0}^{\infty}$ defined 'inductively' by $b_n = \begin{cases} b & \text{if } n = 0 \\ \varphi(b_{n-1}) & \text{if } n \geq 1 \end{cases}$ is an infinite sequence in A with no repeated terms.

(In fact, for each $m \in \mathbb{N}$, $b_m = \underbrace{(\varphi \circ \varphi \circ \ldots \circ \varphi \circ \varphi)}_{m \text{ times}}(b)$.)

The function $f : \mathbb{N} \to A$ defined by $f(n) = b_n$ for any $n \in \mathbb{N}$ is an injective function. Hence $\mathbb{N} \lesssim A$.

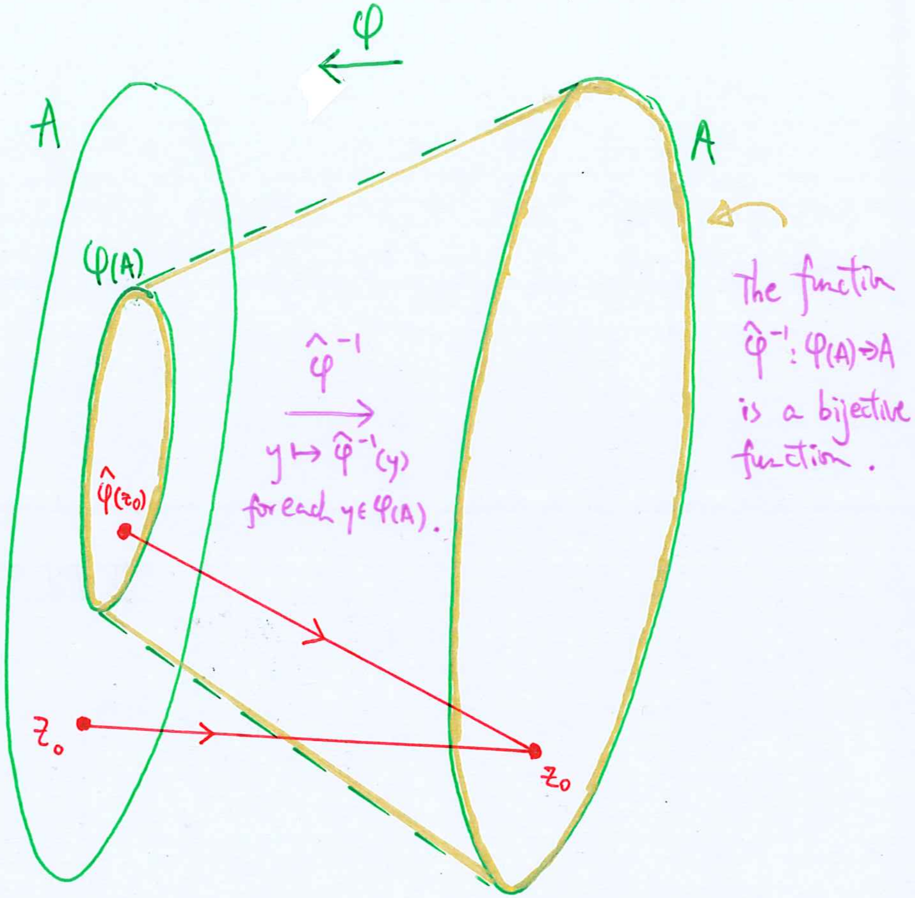
Assumption.

There exists some function $\varphi : A \to A$ such that φ is injective and φ is not surjective.



The function $\hat{\varphi} : A \to \varphi(A)$ defined by $\hat{\varphi}(x) = \varphi(x)$ for any $x \in A$ is a bijective function.

Conclusion.



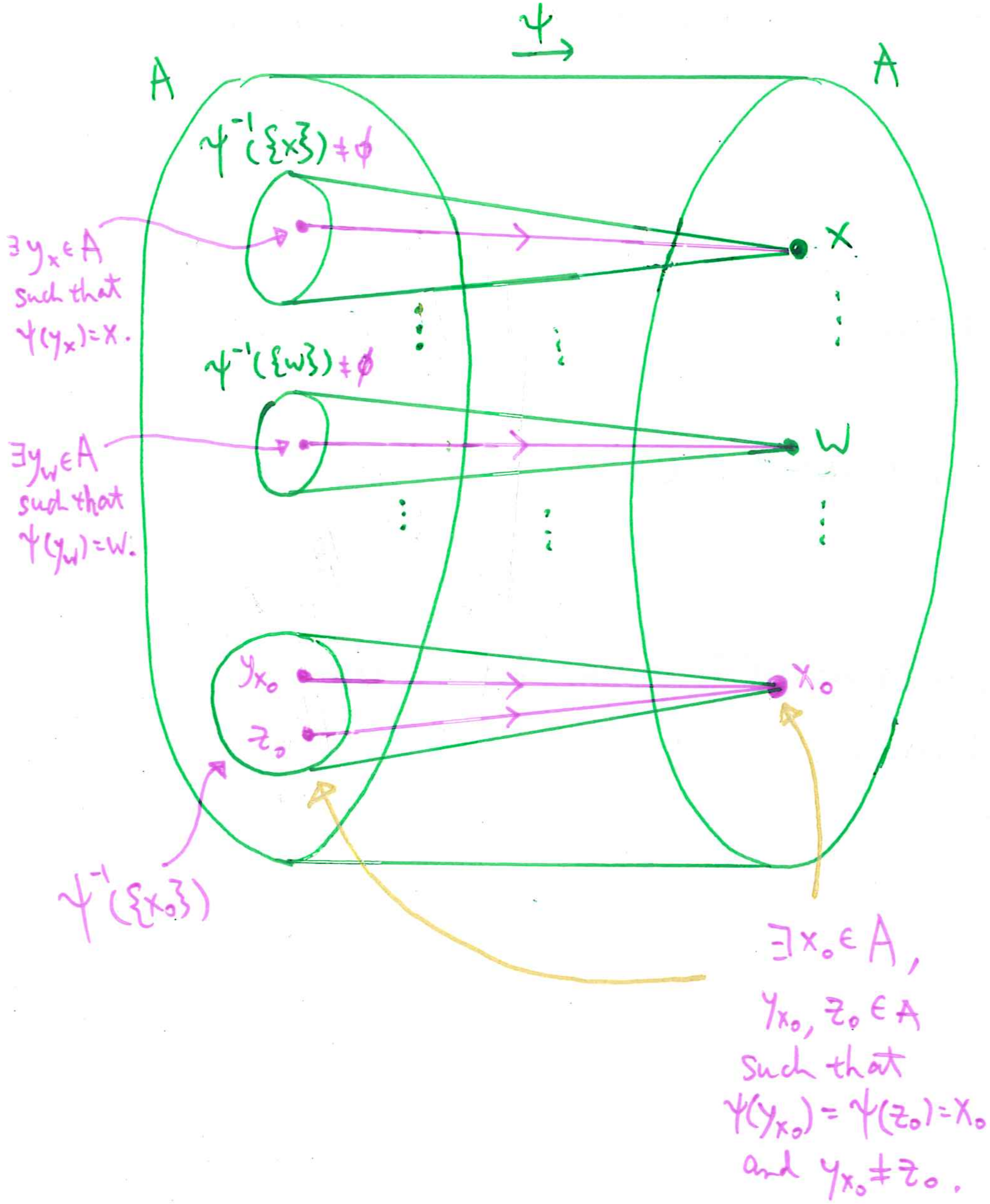
The function $\psi : A \to A$ defined by

$$\psi(y) = \begin{cases} \hat{\varphi}^{-1}(y) & \text{if } y \in \varphi(A) \\ z_0 & \text{if } y \notin \varphi(A) \end{cases}$$

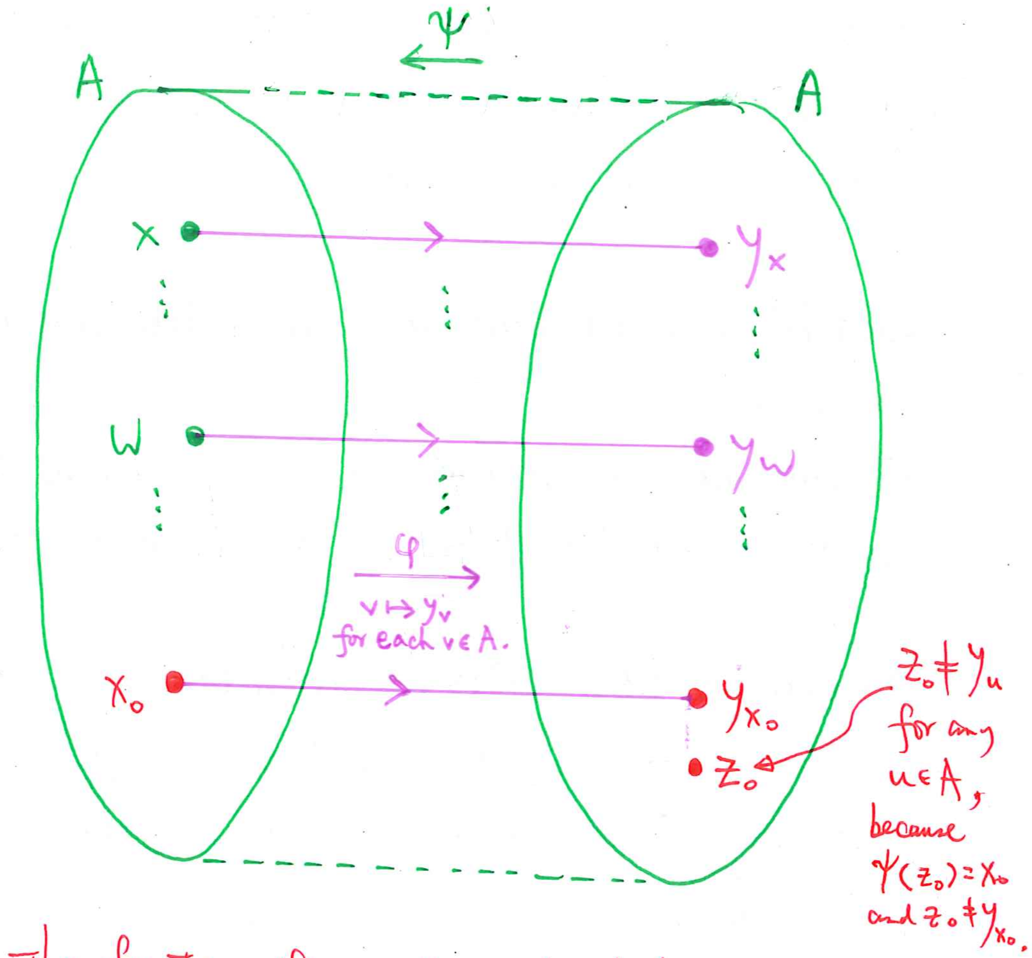
is surjective but not injective.

Assumption.

There exits some function $\psi : A \to A$ such that ψ is surjective and ψ is not injective.



Conclusion.



The function $\varphi : A \to A$ defined by
$\varphi(x) = y_x$ for any $x \in A$
is injective and not surjective.