MATH1050 Zermelo-Fraenkel Axioms with the Axiom of Choice

1. **Terminologies.**

   (a) The only (mathematical) objects in the discourse are sets:

   'all things are sets'.

   (b) Undefined terms/phrases: 'sets', 'belonging to ... (as an element)'.

   All other terms are defined in terms of these.

   We write $x \in S$ exactly when the set x belongs to the set S; equivalently we say the set x is an element of the set S.

   We write $x \notin S$ exactly when it is not true that $x \in S$.

   (c) Common sense: equality '='.

   This symbol possesses these properties (as in 'usual usage'):

   i. *For any object x, the statement '$x = x$' is true.*

   ii. *For any object x, y, if $x = y$ then $y = x$.*

   iii. *For any object x, y, z if ($x = y$ and $y = z$) then $x = z$.*

2. **Axiom of Existence, Axiom of Extention and Axiom of Specification.**

   (1) **Axiom of Existence**.

   There is a set.

   (2) **Axiom of Extension**.

   Let A, B be sets. The statements below are equivalent to each other:

   ($\sharp_2$) $A = B$.

   ($\flat_2$) *For any object x, ($x \in A$ iff $x \in B$).*

   (3) **Axiom of Specification**.

   Suppose A is a set. Suppose '$P(u)$' is a predicate with variable u. Then there exists a set B such that, for any object x, the statements below are equivalent:

   ($\sharp_3$) $x \in B$.

   ($\flat_3$) *$x \in A$ and the statement '$P(x)$' is true.*

   Remark on notations. The set B in this statement is denoted by $\{x \in A : P(x)\}$.

   Remark. In this discourse, there is only one 'method of specification'. We are allowed to construct the set $\{x \in A : P(x)\}$ out of a given set A and a given predicate $P(x)$. Given a predicate $Q(x)$, We are not allowed to immediately regard the object $\{x : Q(x)\}$ as a set, unless further axioms allow us to do so for that specific $Q(x)$.

3. **Existence of the empty set and non-existence of the 'universal set'.**

   Theorem 1.

   There exists some unique set E such that $((\forall x)(x \notin E))$.

   Remark. Why such a 'theorem'? We want our theory not to be void: there is some object to talk about, and one such a set is the 'empty set'. All other objects in our theory is built upon this set. There is no guarantee of the existence and the uniqueness of the empty set until proven; it is not enough to say that the empty set is a set which contains no element.

   Outline of the proof of Theorem 1.

   - *Existence argument.*

     Pick some set A, according to the Axiom of Existence.

     Define $E = \{x \in A : x \neq x\}$, according to the Axiom of Specification.

     Verify $((\forall x)(x \notin E))$. (Fill in the details as an exercise.)

   - *Uniqueness argument.*

     Pick any set B. Suppose $(\forall x)(x \notin B)$. Apply the Axiom of Extension to verify $B = E$. (Fill in the detail as an exercise.)

Remark. The existence argument in the proof of Theorem (1) is the only occassion we need the Axiom of Existence.

Remark on terminology and notation. From now on, we will refer to the set E for which the statement '$(\forall x)(x \notin E)$' is true as the empty set. We will denote it by $\emptyset$.

Theorem 2.

Let A be a set. There exists some set B such that $B \notin A$.

Remark. It follows that there is no 'universal set' that contains every conceivable objects as its elements.

Proof of Theorem 2.

Let A be a set. Define $B = \{x \in A : x \notin x\}$, according to the Axiom of Specification.

Suppose it were true that $B \in A$. If $B \in B$ then $B \notin B$. If $B \notin B$ then $B \in B$. In each case, contradiction arises.

It follows that the assumption $B \in A$ is false. Hence $B \notin A$.

Remark. Russell's Paradox is resolved.

4. **Subsets, intersections and complements.**

   Definitions.

   (a) *Let A, B be sets. A is said to be a subset of B if the statement '$(\forall x)$(if $x \in A$ then $x \in B$)' holds. We write $A \subset B$.*

   (b) *Let A, B be sets. The intersection of A and B is defined to be the set $\{x \in A : x \in B\}$. It is denoted by $A \cap B$.*

   (c) *Let A, B be sets. The complement of B and A is defined to be the set $\{x \in A : x \notin B\}$.*

   Remark. According to the Axiom of Specification, the respective definitions for intersection and complement make sense.

   Theorem 3.

   The following statements hold:

   (3a) *Let A be a set. $\emptyset \subset A$, $A \subset A$, $A \cap \emptyset = \emptyset$, $A \cap A = A$, $A \backslash \emptyset = A$, and $A \backslash A = \emptyset$.*

   (3b) *Let A, B, C be sets. $A \cap B = B \cap A$, and $(A \cap B) \cap C = A \cap (B \cap C)$.*

5. **Axiom of Pairing and Axiom of Union.**

   (4) **Axiom of Pairing**.

   For any objects x, y, there exists a set A such that $x \in A$ and $y \in A$.

   (5) **Axiom of Union**.

   For any set A, there exists a set B such that, for any object x, (if ($x \in S$ for some $S \in A$) then $x \in B$).

6. **Singletons, 'double-tons' and unions.**

   Theorem 4.

   For any objects x, y, there exists some unique set A such that $((\forall z)((z \in A)$ iff $(z = x$ or $z = y)))$.

   Remark on notation. This set A is denoted by $\{x, y\}$. We need the Axiom of Pairing for its construction.

   Theorem 5.

   For any object x, there exists some unique set A such that $((\forall z)((z \in A)$ iff $(z = x)))$.

   Remark on notation. This set A is denoted by $\{x\}$. Such a set is called a singleton. The proof of this result relies on the previous result.

   Theorem 6.

   $\emptyset \neq \{\emptyset\}$.

   Theorem 7.

   Let A, B be sets. There exists some unique set C such that $((\forall x)(x \in C$ iff $(x \in A$ or $x \in B)))$.

   Remark on notation. The set C in this statement is called the union of A, B, and is denoted by $A \cup B$. We also abuse notation to write $A \cup B = \{x \mid x \in A$ or $x \in B\}$. We need the Axiom of Union for its contruction.

7. **Theorem 8.**

   The following statements hold:

   (a) *Let A be a set. $A \cup A = A$, and $A \cup \emptyset = A$.*

   (b) *Let A, B, C be sets. $A \cup B = B \cup A$, and $(A \cup B) \cup C = A \cup (B \cup C)$.*

   (c) *Let A, B, C be sets. $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$, and $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.*

   (d) *Let A, B, C be sets. $(A \cap B) \backslash C = (A \backslash C) \cap (B \backslash C)$, and $(A \cup B) \backslash C = (A \backslash C) \cup (B \backslash C)$.*

   (e) *Let A, B, C be sets. $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$, and $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.*

   Remark. The basic results (such as the Distributive Laws, De Morgan's Laws) concerned with the set operations 'intersection, union, complement' hold.

8. **Axiom of Power, Axiom of Choice, Axiom of Infinity and Axiom of Substitution.**

   (6) **Axiom of Power.**

   The totality of all subsets of any given set constitutes a set.

   (7) **Axiom of Choice.**

   The cartesian product of any non-empty family of non-empty sets is non-empty.

   (8) **Axiom of Infinity.**

   There is a successor set.

   (9) **Axiom of Substitution.**

   Suppose A is a set. Suppose '$S(u, v)$' is a predicate (with two variables u, v) which satisfies the condition that for each object x, there exists an object y such that the statement '$S(x, y)$' is true. Then there exists a set B such that, for any object y, the statements below are equivalent to each other:

   ($\sharp_9$) $y \in B$.

   ($\flat_9$) *The statement '$S(x, y)$' is true for some $x \in A$.*

9. **Power sets, cartesian products, relations and functions, families.**

   Definition.

   Let A be a set. The set of all subsets of A is called the power set of A.

   Remark on notation. We denote the power set of A by $\mathfrak{P}(A)$. We abuse notations to write $\mathfrak{P}(A) = \{S \mid S \subset A\}$. We need the Axiom of Power for this construction.

   Definition.

   Let x, y be objects. We define the ordered pair (x, y) to be the set $\{\{x\}, \{x, y\}\}$.

   Remark. To make sense of (x, y), we need Theorem 4 and Theorem 5.

   Definition.

   Let x, y, z be objects. We define the ordered triple (x, y, z) to be the ordered pair $((x, y), z)$.

   Theorem 9.

   Let x, y, z, u, v, w be objects. The following statements hold:

   (9a) *$(x, y) = (u, v)$ iff ($x = u$ and $y = v$).*

   (9b) *$(x, y, z) = (u, v, w)$ iff ($x = u$ and $y = v$ and $z = w$).*

   Theorem 10.

   Let A, B be sets. There exists some unique set C such that $((\forall z)(z \in C)$ iff ($z = (x, y)$ for some $x \in A$, $y \in B$)).

   Remark on notation. The set C in this statement is called the Cartesian product of A and B, and is denoted by $A \times B$. We abuse notations to write $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$. To construct this set, we need the help of power sets; hence indirectly we need the Axiom of Power.

   We define the notions of relations, functions, equivalence relations, and partial orderings in terms of all the above, as we did casually.

   Definitions.

   - *Let H, K, L be sets. (H, K, L) is said to be a relation from H to K with graph L if $L \subset H \times K$.*

- *Let A, B, F be sets. Suppose (A, B, F) is a relation from A to B with graph F. (A, B, F) is said to be a function from A to B with graph F if the statements $(E), (U)$ below hold:*

  (E): *For any $x \in A$, there exists some $y \in B$ such that $(x, y) \in F$.*

  (U): *For any $x \in A$, for any $y, z \in B$, if $(x, y) \in F$ and $(x, z) \in F$ then $y = z$.*

We recover most of the basic results we have proved casually. However, at some stage, the Axiom of Choice will have to creep in for the justification of some seemingly obvious statements. To make sense of the statement of the Axiom of Choice, we introduce the notion of family and some related definitions.

Definition.

Let I, S be sets. A familiy in S indexed by I is a function from I to S.

Remark on notation. For such a family, say, $x : I \longrightarrow S$, we usually write $x(\beta)$ as x_β, and refer to the family as $\{x_\alpha\}_{\alpha \in I}$.

Definitions.

Let M, I be sets, and $\{S_\alpha\}_{\alpha \in I}$ be a family in $\mathfrak{P}(M)$.

- *We define the intersection of $\{S_\alpha\}_{\alpha \in I}$ to be the set $\{x \in M : x \in S_\alpha$ for any $\alpha \in I\}$.*
- *We define the union of $\{S_\alpha\}_{\alpha \in I}$ to be the set $\{x \in M : x \in S_\alpha$ for some $\alpha \in I\}$.*
- *We define the Cartesian product of $\{S_\alpha\}_{\alpha \in I}$ to be the set $\{\varphi \in \mathsf{Map}(I, M) : \varphi(\alpha) \in I\}$. ($\mathsf{Map}(I, M)$ is the set of all functions from I to M.)*

10. **From the empty set to all the number systems.**

Definition.

Let x be a set. The set $x \cup \{x\}$ is called the successor of x, and is denoted by x^+.

Remark. $\emptyset^+ = \{\emptyset\}$. Write $0 = \emptyset$, $1 = \{\emptyset\}$. $(\emptyset^+)^+ = \{0, 1\}$. Write $2 = \{0, 1\}$. $((\emptyset^+)^+)^+ = \{0, 1, 2\}$. Write $3 = \{0, 1, 2\}$. $(((\emptyset^+)^+)^+)^+ = \{0, 1, 2, 3\}$. *Et cetera.*

Definition.

Let A be a set. A is said to be a successor set if, $\emptyset \in A$ and (for any $x \in A$, $x^+ \in A$).

Theorem 11.

There is a unique 'smallest' successor set $\mathbb{N}$ in the following sense:

(a) *$\mathbb{N}$ is a successor set.*

(b) *For any successor set M, $\mathbb{N} \subset M$.*

Remark on notation. $\mathbb{N}$ is called the set of all natural numbers, and its elements are called natural numbers. By definition, $\emptyset \in \mathbb{N}$. We write $0 = \emptyset$ and call it zero. We write $1 = \emptyset^+$ and call it one. *Et cetera.* The existence argument in the proof of Theorem 11 relies on the Axiom of Infinity.

Theorem 12.

Peano's Axioms hold for $\mathbb{N}$, in the sense that the following statements hold:

(P1) $0 \in \mathbb{N}$.

(P2) *For any $n \in \mathbb{N}$, there is some unique $n^+ \in \mathbb{N}$.* (n^+ is called the successor of n.)

(P3) *For any $m, n \in \mathbb{N}$, if $m^+ = n^+$ then $m = n$.*

(P4) *For any $n \in \mathbb{N}$, $0 \neq n^+$.*

(P5) *Let S be a subset of $\mathbb{N}$. Suppose $0 \in S$. Also suppose that for any $n \in \mathbb{N}$, if $n \in S$ then $n^+ \in S$. Then $S = \mathbb{N}$. (This is the Principle of Mathematical Induction.)*

With the help of the Principle of Mathematical Induction, we define addition in $\mathbb{N}$ 'inductively'. With addition in $\mathbb{N}$ and again with the help of the Principle of Mathematical Induction, we define multiplication in $\mathbb{N}$. The usual ordering of $\mathbb{N}$ is 'encoded' in the notion of successors. Hence we have the natural number system.

With the natural number system, we may successively construct the integer system, the rational number system, the real number system and the complex number system, with the help of functions and relations.