MATH1050 Cantor's Theorem and its consequences

1. Recall:
   (a) **Definition.**
   Let A, B be sets. The set $\mathsf{Map}(A, B)$ is defined to be the **set of all functions from** A **to** B.
   Remark. $\mathsf{Map}(\mathbb{N}, B)$ is the set of all infinite sequences in B: each $\varphi \in \mathsf{Map}(\mathbb{N}, B)$ is the infinite sequence $(\varphi(0), \varphi(1), \varphi(2), ..., \varphi(n), \varphi(n+1), ...)$.
   (b) **Example** (ϵ). *Let A be a set. $\mathfrak{P}(A) \sim \mathsf{Map}(A, \{0,1\})$.*
   (c) **Theorem (VI).** *There is no surjective function from $\mathbb{N}$ to $\mathsf{Map}(\mathbb{N}, \{0,1\})$.*
   (d) **Corollary (VII).** *There is no bijective function from $\mathbb{N}$ to $\mathsf{Map}(\mathbb{N}, \{0,1\})$. (Hence $\mathbb{N} \not\sim \mathsf{Map}(\mathbb{N}, \{0,1\})$.)*
   (e) **Theorem (VIII).** *Let A be a set. $A \not\sim \mathsf{Map}(A, \{0,1\})$. $A \not\sim \mathfrak{P}(A)$.*

2. **Theorem (XIII). (Baby version of Cantor's Theorem.)**
   $\mathbb{N} < \mathsf{Map}(\mathbb{N}, \{0,1\})$.
   Proof.
   By Corollary (VII), $\mathbb{N} \not\sim \mathsf{Map}(\mathbb{N}, \{0,1\})$. We now prove that $\mathbb{N} \lesssim \mathsf{Map}(\mathbb{N}, \{0,1\})$:
   - For any $m \in \mathbb{N}$, define $\delta_m : \mathbb{N} \longrightarrow \{0,1\}$ by
   $$\delta_m(n) = \begin{cases} 1 & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m \end{cases}$$
   Define $\Delta : \mathbb{N} \longrightarrow \mathsf{Map}(\mathbb{N}, \{0,1\})$ by $\Delta(n) = \delta_n$ for any $n \in \mathbb{N}$.
   Δ is an injective function. (Why?)
   Hence $\mathbb{N} \lesssim \mathsf{Map}(\mathbb{N}, \{0,1\})$.

   We now have $\mathbb{N} \lesssim \mathsf{Map}(\mathbb{N}, \{0,1\})$ and $\mathbb{N} \not\sim \mathsf{Map}(\mathbb{N}, \{0,1\})$. It follows that $\mathbb{N} < \mathsf{Map}(\mathbb{N}, \{0,1\})$.

3. **Theorem (XIV). (Cantor's Theorem.)**
   Let A be a set. $A < \mathsf{Map}(A, \{0,1\})$. $A < \mathfrak{P}(A)$.
   Proof.
   Let A be a set. By Theorem (VIII), $A \not\sim \mathsf{Map}(A, \{0,1\})$. We generalize the argument for Theorem (XIII) to prove that $A \lesssim \mathsf{Map}(A, \{0,1\})$:
   - Recall that for any $x \in A$, the function $\chi^A_{\{x\}} : A \longrightarrow \{0,1\}$ is given by
   $$\chi^A_{\{x\}}(y) = \begin{cases} 1 & \text{if} \quad y = x \\ 0 & \text{if} \quad y \neq x \end{cases}$$
   ($\chi^A_{\{x\}}$ is the characteristic function of $\{x\}$ in A.)
   Define the function $\Delta : A \longrightarrow \mathsf{Map}(A, \{0,1\})$ by $\Delta(x) = \chi^A_{\{x\}}$ for any $x \in A$. Δ is an injective function from A to $\mathsf{Map}(A, \{0,1\})$. (Why?) Hence $A \lesssim \mathsf{Map}(A, \{0,1\})$.

   We now have $A \lesssim \mathsf{Map}(A, \{0,1\})$ and $A \not\sim \mathsf{Map}(A, \{0,1\})$.
   It follows that $A < \mathsf{Map}(A, \{0,1\})$. Since $\mathfrak{P}(A) \sim \mathsf{Map}(A, \{0,1\})$, we have $A < \mathfrak{P}(A)$. (Why?)

4. **Question.** *Note that $\mathbb{Q} \lesssim \mathbb{R}$. Is it true that $\mathbb{Q} \sim \mathbb{R}$, or that $\mathbb{Q} < \mathbb{R}$?*

   Lemma (XV).
   Let A, B, C be sets. Suppose $A \lesssim B$ and $B \lesssim C$. Also suppose $A < B$ or $B < C$. Then $A < C$.
   Proof.
   Let A, B, C be sets. Suppose $A \lesssim B$ and $B \lesssim C$. Also suppose $A < B$ or $B < C$.
   Since $A \lesssim B$ and $B \lesssim C$, we have $A \lesssim C$.
   Since $A < B$ or $B < C$, we have $A \not\sim B$ or $B \not\sim C$. We verify that $A \not\sim C$:

- Suppose it were true that $A \sim C$. Then $C \lesssim A$.

  Since $B \lesssim C$ and $C \lesssim A$, we would have $B \lesssim A$. Then, since $A \lesssim B$ and $B \lesssim A$, we would have $A \sim B$ according to the Schröder-Bernstein Theorem.

  Since $C \lesssim A$ and $A \lesssim B$, we would have $C \lesssim B$. Then, since $B \lesssim C$ and $C \lesssim B$, we would have $B \sim C$ according to the Schröder-Bernstein Theorem.

  Hence $A \sim B$ and $B \sim C$. But by assumption, $A \not\sim B$ or $B \not\sim C$. Contradiction arises. Hence $A \not\sim C$ in the first place.

Then, since $A \lesssim C$ and $A \not\sim C$, we have $A < C$.

Theorem (XVI).

$\mathbb{N} < [0,1]$. $\mathbb{N} < \mathbb{R}$. $\mathbb{Q} < \mathbb{R}$.

Proof.

$\mathbb{N} \lesssim \mathsf{Map}(\mathbb{N}, \{0,1\}) \lesssim \mathsf{Map}(\mathbb{N}, [\![0,9]\!]) \sim [0,1] \sim \mathbb{R}$.

Also, $\mathbb{N} < \mathsf{Map}(\mathbb{N}, \{0,1\})$.

Then, by Lemma (XV), $\mathbb{N} < [0,1]$ and $\mathbb{N} < \mathbb{R}$.

Since $\mathbb{Q} \sim \mathbb{N}$, we also have $\mathbb{Q} < \mathbb{R}$.

Remark. Hence there are much much more real numbers than there are rational numbers.

5. **Question.** *Why are'Venn diagram arguments' not good enough?*

Theorem (XVII.)

There exists some set T such that $S < T$ for any subset S of $\mathbb{R}^2$.

Proof.

Define $T = \mathfrak{P}(\mathbb{R})$.

Pick any subset S of $\mathbb{R}^2$. We have $S \lesssim \mathbb{R}^2 \sim \mathbb{R}$. By Cantor's Theorem, $\mathbb{R} < \mathfrak{P}(\mathbb{R}) = T$. Then by Lemma (XV), we have $S < T$.

Remark.

When we draw a Venn diagram for a set, say, A, we are 'identifying' the set A with some subset, say, B, of $\mathbb{R}^2$, in the sense that the elements of A are 'identified' as the points in B, via some bijective function from A to B. This bijective function guarantees that distinct elements of A are identified as distinct points of B. So we are implicitly assuming that there is an injective function from A to $\mathbb{R}^2$.

But now we know that there are sets which are too 'large' to be draw in a Venn diagram.

6. **Question.** *Is there any 'universal set', which contains every conceivable object as its element?*

Theorem (XVIII).

Denote $\{x \mid x = x\}$ by U. The mathematical object U is not a set.

Proof.

Suppose U were a set. Then, by Cantor's Theorem, $U < \mathsf{Map}(U, \{0,1\})$.

For any $\varphi \in \mathsf{Map}(U, \{0,1\})$, we would have $\varphi = \varphi$, and hence $\varphi \in U$.

It would follow that $\mathsf{Map}(U, \{0,1\}) \subset U$.

Then $\mathsf{Map}(U, \{0,1\}) \lesssim U$. Therefore $U < \mathsf{Map}(U, \{0,1\}) \lesssim U$.

By Lemma (XV), $U < U$. In particular, $U \not\sim U$. There would be no bijective function from U to U. But id_U is a bijective function from U to U. Contradiction arises.

Hence U is not a set in the first place.

Remark. Hence if we insist Cantor's Theorem to be a true statement, then there is no such thing as a 'universal set'. This is known as **Cantor's Paradox**.