1. **Theorem (XI). (Schröder-Bernstein Theorem.)**

   Let A, B be sets. Suppose $A \lesssim B$ and $B \lesssim A$. Then $A \sim B$.

   We postpone the proof of the Schröder-Bernstein Theorem. For the moment, we take for granted the validity of this result and see its applications in various examples.

   Remark.

   What is so special about Schröder-Bernstein Theorem?

   Recall the definition for the notion of equal cardinality:

   $A \sim B$ iff there is a bijective function from A to B.

   Imagine we want to verify that two given sets, say, A, B, are of equal cardinality. If we adhere to definition, we have to write down a relation, say, h, from A to B and verify that h is a bijective function. It is very often no easy task, even when A, B are not very complicated sets. (Recall how we verify $[0,1] \sim [0,1)$ by constructing a bijective function from $[0,1]$ to $[0,1)$, in the Handout *Sets of equal cardinality.* The difficulty in this specific example arises from the fact that we are not used to thinking of functions which do not look 'nice': in this case a function is not continuous at many points is involved. But this is the price for ensuring that what we write down is a bijective function.) The Schröder-Bernstein Theorem offers a way out: to verify $A \sim B$, it suffices to give two injective functions, one from A to B and the other from B to A, instead of one bijective function from A to B. In many situations, the former is much easier.

2. **Example (A).**

   Another argument for $\mathbb{N} \sim \mathbb{N}^2$.

   - Define $f : \mathbb{N} \longrightarrow \mathbb{N}^2$ by $f(x) = (x, 0)$ for any $x \in \mathbb{N}$.
     f is injective. (Exercise.)
     It follows that $\mathbb{N} \lesssim \mathbb{N}^2$.
   - Define $g : \mathbb{N}^2 \longrightarrow \mathbb{N}$ by $g(x, y) = 2^x \cdot 3^y$ for any $x, y \in \mathbb{N}$.
     g is injective. (Exercise.)
     It follows that $\mathbb{N}^2 \lesssim \mathbb{N}$.
   - Now we have $\mathbb{N} \lesssim \mathbb{N}^2$ and $\mathbb{N}^2 \lesssim \mathbb{N}$.
     According to the Schröder-Bernstein Theorem, $\mathbb{N} \sim \mathbb{N}^2$.

3. **Example (B).**

   A simple argument for $\mathbb{N} \sim \mathbb{Q}$.

   - We have $\mathbb{N} \subset \mathbb{Q}$. Then $\mathbb{N} \lesssim \mathbb{Q}$.
   - We have $\mathbb{Q} \lesssim \mathbb{Z}^2 \sim \mathbb{N}^2 \sim \mathbb{N}$. (How comes $\mathbb{Q} \lesssim \mathbb{Z}$? Fill in the detail.) Then $\mathbb{Q} \lesssim \mathbb{N}$.
   - Now $\mathbb{N} \lesssim \mathbb{Q}$ and $\mathbb{Q} \lesssim \mathbb{N}$.
     According to the Schröder-Bernstein Theorem, $\mathbb{N} \sim \mathbb{Q}$. We also have $\mathbb{N} \sim \mathbb{Z}$ and $\mathbb{Z} \sim \mathbb{Q}$.

   Remark. Hence there are as many natural numbers as there are integers or rational numbers.

4. **Example (C).**

   Let S, T be subsets of $\mathbb{R}$. Suppose S contains as a subset some interval with two or more points. Suppose T contains as a subset some interval with two or more points. Then $S \sim T$.

   We illustrate the validity of this statement through some simple examples.

   (C1) $(0,1) \sim [0,1]$.
   Justification:
   - Define $f : (0,1) \longrightarrow [0,1]$ by $f(x) = x$ for any $x \in (0,1)$.
   - Define $g : [0,1] \longrightarrow (0,1)$ by $g(x) = \dfrac{x+1}{3}$ for any $x \in [0,1]$.
   - f, g are injective functions. (Exercise.)
     Hence $(0,1) \lesssim [0,1]$ and $[0,1] \lesssim (0,1)$.
     According to the Schröder-Bernstein Theorem, $(0,1) \sim [0,1]$.

   (C2) $[-1,1] \sim \mathbb{R}$.
   Justification:
   - Define $f : [-1,1] \longrightarrow \mathbb{R}$ by $f(x) = x$ for any $x \in [-1,1]$.

- Define $g : \mathbb{R} \longrightarrow [-1, 1]$ by $g(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ for any $x \in \mathbb{R}$.
- f, g are injective. (Exercise.)
  Hence $[-1, 1] \lesssim \mathbb{R}$ and $\mathbb{R} \lesssim [-1, 1]$.
  According to the Schröder-Bernstein Theorem, $[-1, 1] \sim \mathbb{R}$.

With a similar argument we can deduce that $I \sim J$ whenever I, J are intervals with at least two points. (Provide the detail.)

Remarks.

- How to prove $[-1, 1] \cup (2, 3) \sim [-2, 0] \cup [1, 4)$?
- How about $[1, 2] \cup \mathbb{Q} \sim (0.01, 0.09) \cup (0.1, 0.99) \cup \mathbb{N}$?
- How to prove the original statement for the general situation?

5. **Example (D).**

Recall that $\mathsf{Map}(\mathbb{N}, [\![0, 9]\!])$ is the set of all infinite sequences in $[\![0, 9]\!]$.

We argue for $[0, 1] \sim \mathsf{Map}(\mathbb{N}, [\![0, 9]\!])$:

- For each $r \in [0, 1]$, choose one decimal representation of r and write $r = 0.r_0r_1r_2r_3\cdots$, and then define the infinite sequence $\alpha(r) = (r_0, r_1, r_2, r_3, \cdots)$.
  No two distinct real numbers have the same decimal representation.
  In this way we have defined the injective function $\alpha : [0, 1] \longrightarrow \mathsf{Map}(\mathbb{N}, [\![0, 9]\!])$, given by $r \longmapsto \alpha(r)$ for any $r \in [0, 1]$.
  Therefore $[0, 1] \lesssim \mathsf{Map}(\mathbb{N}, [\![0, 9]\!])$.
- Define the function $\rho : \mathsf{Map}(\mathbb{N}, [\![0, 9]\!]) \longrightarrow [0, 1]$ by

$$\rho(\{a_n\}_{n=0}^{\infty}) = 0.a_05a_15a_25a_35\cdots \qquad \text{for any } \{a_n\}_{n=0}^{\infty} \in \mathsf{Map}(\mathbb{N}, [\![0, 9]\!]).$$

  ρ is injective. (Exercise.)
  (We can use any one of $1, 2, \cdots, 8$ in place of 5 in the construction of such an injective function.)
  Therefore $\mathsf{Map}(\mathbb{N}, [\![0, 9]\!]) \lesssim [0, 1]$.
- According to the Schröder-Bernstein Theorem, $[0, 1] \sim \mathsf{Map}(\mathbb{N}, [\![0, 9]\!])$.

Consequences:

(D1) $[0, 1] \sim \mathsf{Map}(\mathbb{N}, [\![0, 9]\!]) \sim (\mathsf{Map}(\mathbb{N}, [\![0, 9]\!]))^2 \sim [0, 1]^2$.
Hence there are as many points in the line segment $[0, 1]$ as there are in the square $[0, 1]^2$.

(D2) $\mathbb{R} \sim [0, 1] \sim [0, 1]^2 \sim \mathbb{R}^2 \sim \mathbb{C}$.
There are as many real numbers as there are complex numbers.

(D3) Applying mathematical induction, we have $\mathbb{R} \sim \mathbb{R}^n$, $\mathbb{C} \sim \mathbb{C}^n$ for any $n \in \mathbb{N} \backslash \{0\}$.

Remarks.

(1) Now it remains to see compare the 'relative sizes' of $\mathbb{Q}$ and $\mathbb{R}$.

(2) What is the significance of $\mathbb{R} \sim \mathbb{R}^n$ for any $n \in \mathbb{N} \backslash \{0\}$?
It is that we cannot define 'dimension' by simply comparing the 'relative sizes' of sets. This surprised Cantor and his contemporaries.

6. **Example (E).**

Let Λ be the set of all lines in $\mathbb{R}^2$. We are going to argue for $\Lambda \sim \mathbb{R}$:

- For each point $(a, b) \in \mathbb{R}^2$, denote by $L_{(a,b)}$ the line given by the equation $y = ax + b$.
  $(a, b) \longmapsto L_{(a,b)}$ defines an injective function from $\mathbb{R}^2$ to Λ.
  Hence $\mathbb{R} \sim \mathbb{R}^2 \lesssim \Lambda$.
- For each line L in $\mathbb{R}^2$, choose one ordered triple (a_L, b_L, c_L) so that L is given by the equation $a_Lx + b_Ly + c_L = 0$.
  $L \longmapsto (a_L, b_L, c_L)$ defines an injective function from Λ to $\mathbb{R}^3$.
  Hence $\Lambda \lesssim \mathbb{R}^3 \sim \mathbb{R}$.
- Now $\mathbb{R} \lesssim \Lambda$ and $\Lambda \lesssim \mathbb{R}$. According to the Schröder-Bernstein Theorem, $\Lambda \sim \mathbb{R}$.

Remark. With similar arguments, we deduce that the set of all planes in $\mathbb{R}^3$, the set of all circles in $\mathbb{R}^2$, the set of all spheres in $\mathbb{R}^3$ et cetera are of cardinality equal to $\mathbb{R}$.

7. **Preparation for a proof of the Schöder-Bernstein Theorem.**

Recall:

(a) **Definition. (Generalized union and generalized intersection.)**

Let M be a set, and $\{S_n\}_{n=0}^{\infty}$ be an infinite sequence of subsets of the set M.

i. *The* **(generalized) intersection of the infinite sequence of subsets** $\{S_n\}_{n=0}^{\infty}$ **of the set** M *is defined to be the set $\{x \in M : x \in S_n$ for any $n \in \mathbb{N}\}$. It is denoted by $\bigcap\limits_{n=0}^{\infty} S_n$.*

ii. *The* **(generalized) union of the infinite sequence of subsets** $\{S_n\}_{n=0}^{\infty}$ **of the set** M *is defined to be the set $\{x \in M : x \in S_n$ for some $n \in \mathbb{N}\}$. It is denoted by $\bigcup\limits_{n=0}^{\infty} S_n$.*

(b) **Theorem (IV). ('Glueing Lemma'.)**

Let A, B be sets. Let $\{C_n\}_{n=0}^{\infty}$, $\{D_n\}_{n=0}^{\infty}$ be infinite sequences of subsets of A, B respectively. Let $\{H_n\}_{n=0}^{\infty}$ be an infinite seuqence of subsets of $A \times B$. Suppose $\{(C_n, D_n, H_n)\}_{n=0}^{\infty}$ is an infinite sequence of bijective functions. Suppose that for any $j, k \in \mathbb{N}$, if $j \neq k$ then $C_j \cap C_k = \emptyset$ and $D_j \cap D_k = \emptyset$. Then $\left(\bigcup\limits_{n=0}^{\infty} C_n, \bigcup\limits_{n=0}^{\infty} D_n, \bigcup\limits_{n=0}^{\infty} H_n\right)$ is a bijective function.

We are going to outline an argument for the Schröder-Bernstein Theorem. The argument will rely on Theorem (IV). The detail in the argument for Theorem (XI) and the proof of the Theorem (IV) are left as exercises.

8. **Outline of an argument for the Schröder-Bernstein Theorem.**

Let A, B be sets. Suppose $A \lesssim B$ and $B \lesssim A$.

Since $A \lesssim B$, there is some injective function from A to B, say, $f : A \longrightarrow B$ with graph F.

Since $B \lesssim A$, there is some injective function from B to A, say, $g : B \longrightarrow A$ with graph G.

When one of f, g is surjective as well, it will be a bijective function as well. Then we will have $A \sim B$ immediately.

From now on, we assume that neither of f, g is surjective.

We are going to construct a bijective function from A to B out of f, g.

[*Idea.* Make use of the non-empty sets $B \backslash f(A)$, $A \backslash g(B)$ and the injective functions f, g to 'break up' A, B respectively into many many pieces. 'Arrange' the 'pieces' 'on the two sides' into many many pairs appropriately, with one bijective function defined by f or g as appropriate 'joining' as its domain and range the two 'pieces' in each pair. 'Glue up' the many many bijective functions together to obtain a bijective function from A to B.]

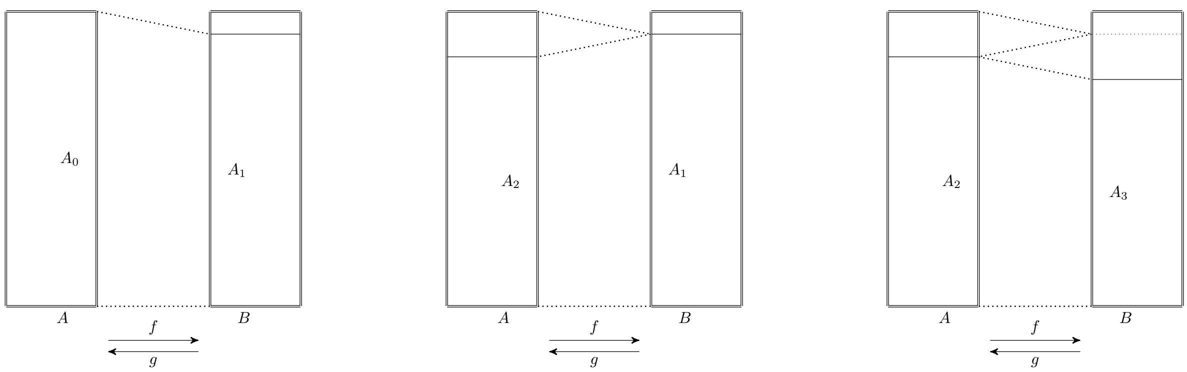
(a) Define $A_0 = A$, $B_0 = B$. For any $n \in \mathbb{N}$, define

$$\begin{array}{rclcrcl} A_{2n+1} & = & f(A_{2n}), & \qquad & A_{2n+2} & = & g(A_{2n+1}), \\ B_{2n+1} & = & g(B_{2n}), & & B_{2n+2} & = & f(B_{2n+1}). \end{array}$$

(So

$$A_1 = f(A_0), \quad A_2 = g(A_1) = g(f(A_0)), \quad A_3 = f(A_2) = f(g(f(A_0))), \quad A_4 = g(A_3) = g(f(g(f(A_0)))),$$

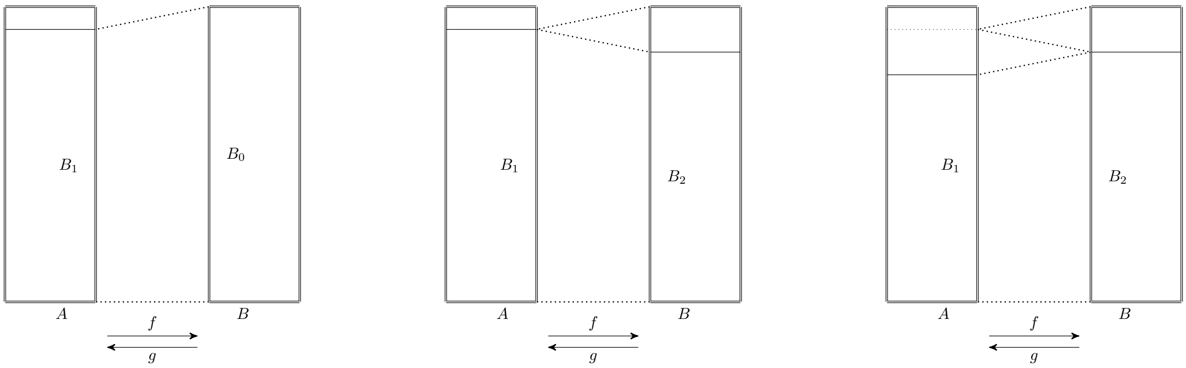
and so forth and so on:



Also,

$$B_1 = g(B_0), \quad B_2 = f(B_1) = f(g(B_0)), \quad B_3 = g(B_2) = g(f(g(B_0))), \quad B_4 = f(B_3) = f(g(f(g(B_0)))),$$

and so forth and so on:



Note that the pictures highlight the injectivity and non-surjectivity of each of f, g.)

(b) We apply mathematical induction to verify the two 'chains of proper subset relations' below:

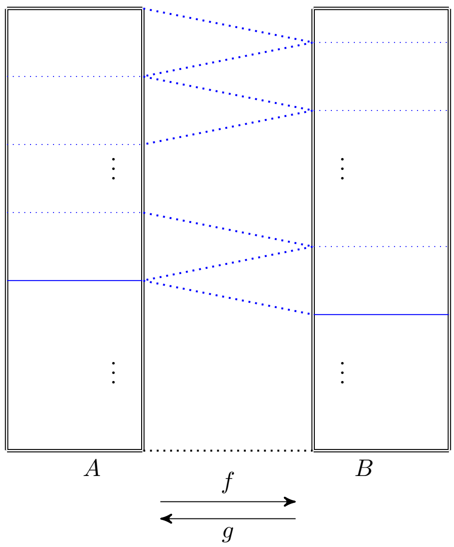
(1) $A = A_0 \supsetneq B_1 \supsetneq A_2 \supsetneq B_3 \supsetneq \cdots \supsetneq A_{2n} \supsetneq B_{2n+1} \supsetneq A_{2n+2} \supsetneq B_{2n+3} \supsetneq \cdots.$

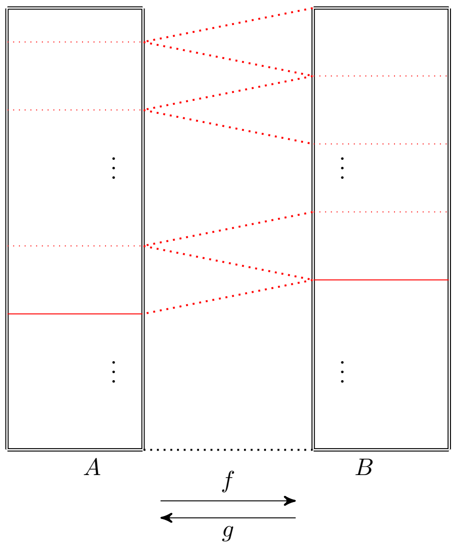
(2) $B = B_0 \supsetneq A_1 \supsetneq B_2 \supsetneq A_3 \supsetneq \cdots \supsetneq B_{2n} \supsetneq A_{2n+1} \supsetneq B_{2n+2} \supsetneq A_{2n+3} \supsetneq \cdots.$

(Below is what we see when we consider the A_n's and the B_m's separately:

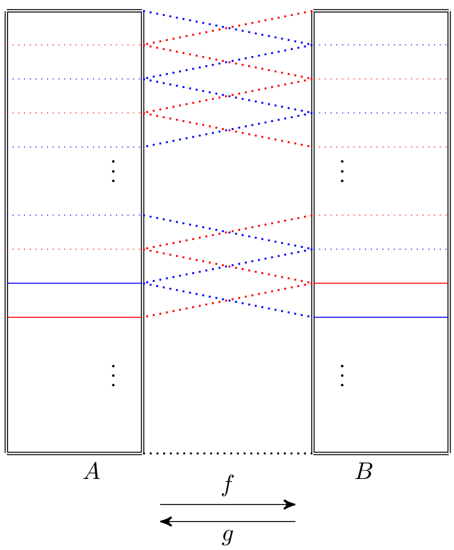
$A_0 \supsetneq A_2 \supsetneq A_4 \supsetneq \cdots \supsetneq A_{2n} \supsetneq \cdots.$ $\quad$ $A_0 \supsetneq B_1 \supsetneq B_3 \supsetneq \cdots \supsetneq B_{2n+1} \supsetneq \cdots.$

$B_0 \supsetneq A_1 \supsetneq A_3 \supsetneq \cdots \supsetneq A_{2n+1} \supsetneq \cdots.$ $\quad$ $B_0 \supsetneq B_2 \supsetneq B_4 \supsetneq \cdots \supsetneq B_{2n} \supsetneq \cdots.$





They combine to give the two 'chains of proper subset relations' described in (1), (2):



It is the injectivity and non-surjectivity of f and g that guarantees each 'proper subset relation' in each 'chain'.)

(c) For any $n \in \mathbb{N}$, define

$$\begin{array}{rclcrcl} C_{2n+1} & = & A_{2n} \backslash B_{2n+1}, & \quad & C_{2n+2} & = & B_{2n+1} \backslash A_{2n+2}, \\ D_{2n+1} & = & B_{2n} \backslash A_{2n+1}, & \quad & D_{2n+2} & = & A_{2n+1} \backslash B_{2n+2}. \end{array}$$

Define $C_0 = \bigcap_{n=0}^{\infty} A_{2n}$, $D_0 = \bigcap_{n=0}^{\infty} B_{2n}$.

We verify the four statements below:

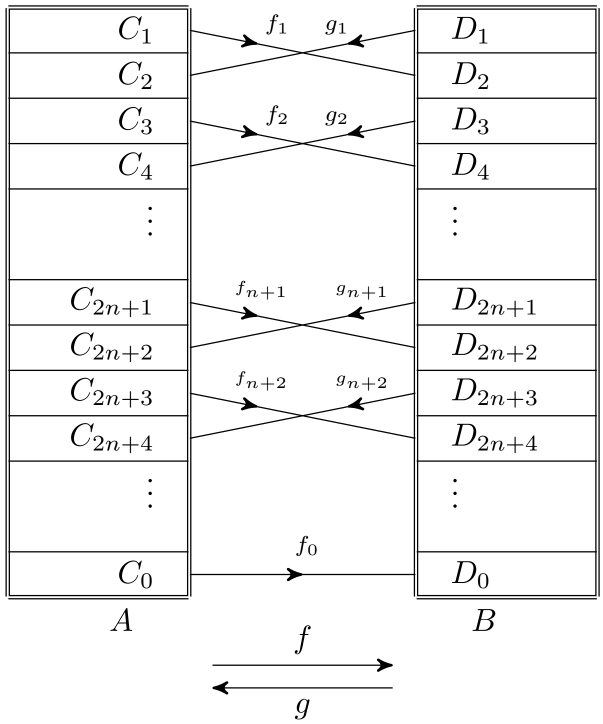
(3) For any $n \in \mathbb{N}$, $C_n \neq \emptyset$ and $D_n \neq \emptyset$.

(4) For any $m, n \in \mathbb{N}$, if $m \neq n$ then $C_m \cap C_n = \emptyset$ and $D_m \cap D_n = \emptyset$.

(5) $C_0 = \bigcap_{n=0}^{\infty} B_{2n+1}$ and $D_0 = \bigcap_{n=0}^{\infty} A_{2n+1}$

(6) $A = \bigcup_{n=0}^{\infty} C_n$, and $B = \bigcup_{n=0}^{\infty} D_n$.

(So f, g combine together to 'split' the sets A, B into the 'chambers' $C_0, C_1, C_2, C_3, \cdots$ and $D_0, D_1, D_2, D_3, \cdots$ respectively:



It will turn out that $C_0 \sim D_0$, and $C_1 \sim D_2$, $C_2 \sim D_1$, $C_3 \sim D_4$, $C_4 \sim D_3$, ..., $C_{2n+1} \sim D_{2n+2}$, $C_{2n+2} \sim D_{2n+1}$, ..., because of the injectivity of f, g.)

(d) Define the relation f_0 by $f_0 = (C_0, D_0, F \cap (C_0 \times D_0))$.

For any $n \in \mathbb{N}$, define the relation f_{n+1} by $f_n = (C_{2n+1}, D_{2n+2}, F \cap (C_{2n+1} \times D_{2n+2}))$.

For any $m \in \mathbb{N}$, define the relation g_{m+1} by $g_{m+1} = (D_{2m+1}, C_{2m+2}, G \cap (D_{2m+1} \times C_{2m+2}))$.

We verify the three statements below:

(7) f_0 is a bijective function.

(8) For any $n \in \mathbb{N}$, f_{n+1} is a bijective function.

(9) For any $m \in \mathbb{N}$, g_{m+1} is a bijective function.

(In fact, $f_0(x) = f(x)$ for any $x \in C_0$. For any $n \in \mathbb{N}$, we have $f_n(x) = f(x)$ for any $x \in C_{2n+1}$. For any $m \in \mathbb{N}$, we have $g_m(y) = g(y)$ for any $y \in D_{2m+1}$.)

(e) Define the function $h : A \longrightarrow B$ by

$$h(x) = \begin{cases} f_0(x) & \text{if} \quad x \in C_0 \\ f_n(x) & \text{if} \quad x \in C_{2n-1} \quad \text{for some} \quad n \in \mathbb{N} \backslash \{0\} \\ {g_m}^{-1}(x) & \text{if} \quad x \in C_{2m} \quad \text{for some} \quad m \in \mathbb{N} \backslash \{0\} \end{cases}$$

We verify that h is a bijective function. (Make use of the Generalized Glueing Lemma.)

It follows that $A \sim B$.