1. **Theorem (XI). (Schröder-Bernstein Theorem.)**

Let A, B be sets. Suppose $A \lesssim B$ and $B \lesssim A$. Then $A \sim B$.

Remark.

What is so special about Schröder-Bernstein Theorem?

- It is usually much more difficult to write down a bijective function than to write down a pair of injective functions. So?

2. **Example (A).**

Another argument for $\mathbb{N} \sim \mathbb{N}^2$.

- Define $f : \mathbb{N} \longrightarrow \mathbb{N}^2$ by $f(x) = (x, 0)$ for any $x \in \mathbb{N}$.

  f is injective. (Exercise.)

  It follows that $\mathbb{N} \lesssim \mathbb{N}^2$.

- Define $g : \mathbb{N}^2 \longrightarrow \mathbb{N}$ by $g(x, y) = 2^x \cdot 3^y$ for any $x, y \in \mathbb{N}$.

  g is injective. (Exercise.)

  It follows that $\mathbb{N}^2 \lesssim \mathbb{N}$.

- Now we have $\mathbb{N} \lesssim \mathbb{N}^2$ and $\mathbb{N}^2 \lesssim \mathbb{N}$.

  According to the Schröder-Bernstein Theorem, $\mathbb{N} \sim \mathbb{N}^2$.

3. **Example (B).**

A simple argument for $\mathbb{N} \sim \mathbb{Q}$.

- We have $\mathbb{N} \subset \mathbb{Q}$. Then $\mathbb{N} \lesssim \mathbb{Q}$.
- We have $\mathbb{Q} \lesssim \mathbb{Z}^2 \sim \mathbb{N}^2 \sim \mathbb{N}$. (Detail for $\mathbb{Q} \lesssim \mathbb{Z}^2$?) Then $\mathbb{Q} \lesssim \mathbb{N}$.
- Now $\mathbb{N} \lesssim \mathbb{Q}$ and $\mathbb{Q} \lesssim \mathbb{N}$.

  According to the Schröder-Bernstein Theorem, $\mathbb{N} \sim \mathbb{Q}$. We also have $\mathbb{N} \sim \mathbb{Z}$ and $\mathbb{Z} \sim \mathbb{Q}$.

Remark.

Hence there are as many natural numbers as there are integers or rational numbers.

4. **Example (C).**

Let S, T be subsets of $\mathbb{R}$.

Suppose S contains as a subset some interval with two or more points.

Suppose T contains as a subset some interval with two or more points.

Then $S \sim T$.

Illustrations through some simple examples.

(C1) $(0,1) \sim [0,1]$.

Justification:

- Define $f : (0,1) \longrightarrow [0,1]$ by $f(x) = x$ for any $x \in (0,1)$.
- Define $g : [0,1] \longrightarrow (0,1)$ by $g(x) = \dfrac{x+1}{3}$ for any $x \in [0,1]$.
- f, g are injective functions. (Exercise.)

  Hence $(0,1) \lesssim [0,1]$ and $[0,1] \lesssim (0,1)$.

  According to the Schröder-Bernstein Theorem, $(0,1) \sim [0,1]$.

Example (C).

Let S, T be subsets of $\mathbb{R}$.

Suppose S contains as a subset some interval with two or more points.

Suppose T contains as a subset some interval with two or more points.

Then $S \sim T$.

· Illustrations through some simple examples.

(C1) $(0,1) \sim [0,1]$.

(C2) $[-1,1] \sim \mathbb{R}$.

Justification:

- Define $f : [-1,1] \longrightarrow \mathbb{R}$ by $f(x) = x$ for any $x \in [-1,1]$.
- Define $g : \mathbb{R} \longrightarrow [-1,1]$ by $g(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ for any $x \in \mathbb{R}$.
- f, g are injective. (Exercise.)

  Hence $[-1,1] \lesssim \mathbb{R}$ and $\mathbb{R} \lesssim [-1,1]$.

  According to the Schröder-Bernstein Theorem, $[-1,1] \sim \mathbb{R}$.

Example (C).

Let S, T be subsets of $\mathbb{R}$. Suppose S contains as a subset some interval with two or more points. Suppose T contains as a subset some interval with two or more points. Then $S \sim T$.

Illustrations through some simple examples.

(C1) $(0,1) \sim [0,1]$.

(C2) $[-1,1] \sim \mathbb{R}$.

With a similar argument we can deduce that $I \sim J$ whenever I, J are intervals with at least two points. (Provide the detail.)

Remarks.

- How to prove $[-1,1] \cup (2,3) \sim [-2,0] \cup [1,4)$?
- How about $[1,2] \cup \mathbb{Q} \sim (0.01, 0.09) \cup (0.1, 0.99) \cup \mathbb{N}$?
- How to prove the statement for the general situation?

5. **Example (D).**

Recall that $\mathsf{Map}(\mathbb{N}, [\![0,9]\!])$ is the set of all infinite sequences in $[\![0,9]\!]$.

We argue for $[0,1] \sim \mathsf{Map}(\mathbb{N}, [\![0,9]\!])$:

- For each $r \in [0,1]$, choose one decimal representation of r and write

$$r = 0.r_0 r_1 r_2 r_3 \cdots ,$$

and then define the infinite sequence

$$\alpha(r) = (r_0, r_1, r_2, r_3, \cdots).$$

No two distinct real numbers have the same decimal representation.

In this way we have defined the injective function

$$\alpha : [0,1] \longrightarrow \mathsf{Map}(\mathbb{N}, [\![0,9]\!]),$$

given by

$$r \longmapsto \alpha(r) \quad \text{for any } r \in [0,1].$$

Therefore $[0,1] \lesssim \mathsf{Map}(\mathbb{N}, [\![0,9]\!])$.

Example (D).

Recall that $\mathsf{Map}(\mathbb{N}, [\![0,9]\!])$ is the set of all infinite sequences in $[\![0,9]\!]$.

We argue for $[0,1] \sim \mathsf{Map}(\mathbb{N}, [\![0,9]\!])$:

- ...

  Therefore $[0,1] \lesssim \mathsf{Map}(\mathbb{N}, [\![0,9]\!])$.

- Define the function

$$\rho : \mathsf{Map}(\mathbb{N}, [\![0,9]\!]) \longrightarrow [0,1]$$

  by

$$\rho(\{a_n\}_{n=0}^{\infty}) = 0.a_0 5 a_1 5 a_2 5 a_3 5 \cdots \qquad \text{for any } \{a_n\}_{n=0}^{\infty} \in \mathsf{Map}(\mathbb{N}, [\![0,9]\!]).$$

  ρ is injective. (Exercise.)

  (We can use any one of $1, 2, \cdots, 8$ in place of 5 in the construction of such an injective function.)

  Therefore $\mathsf{Map}(\mathbb{N}, [\![0,9]\!]) \lesssim [0,1]$.

- According to the Schröder-Bernstein Theorem, $[0,1] \sim \mathsf{Map}(\mathbb{N}, [\![0,9]\!])$.

Example (D).

Recall that $\mathsf{Map}(\mathbb{N}, [\![0,9]\!])$ is the set of all infinite sequences in $[\![0,9]\!]$.

We argue for $[0,1]\sim\mathsf{Map}(\mathbb{N}, [\![0,9]\!])$: $\cdots$

Consequences of $[0,1]\sim\mathsf{Map}(\mathbb{N}, [\![0,9]\!])$:

(D1) $[0,1]\sim\mathsf{Map}(\mathbb{N}, [\![0,9]\!])\sim(\mathsf{Map}(\mathbb{N}, [\![0,9]\!]))^2\sim[0,1]^2$.

Hence there are as many points in the line segment $[0,1]$ as there are in the square $[0,1]^2$.

(D2) $\mathbb{R}\sim[0,1]\sim[0,1]^2\sim\mathbb{R}^2\sim\mathbb{C}$.

There are as many real numbers as there are complex numbers.

(D3) Applying mathematical induction, we have $\mathbb{R}\sim\mathbb{R}^n$, $\mathbb{C}\sim\mathbb{C}^n$ for any $n\in\mathbb{N}\backslash\{0\}$.

Remarks.

(1) Now it remains to see compare the 'relative sizes' of $\mathbb{Q}$ and $\mathbb{R}$.

(2) What is the significance of $\mathbb{R}\sim\mathbb{R}^n$ for any $n\in\mathbb{N}\backslash\{0\}$?

It is that we cannot define 'dimension' by simply comparing the 'relative sizes' of sets.

This surprised Cantor and his contemporaries.

6. **Example (E).**

Let Λ be the set of all lines in $\mathbb{R}^2$. We are going to argue for $\Lambda \sim \mathbb{R}$:

- For each point $(a, b) \in \mathbb{R}^2$, denote by $L_{(a,b)}$ the line given by the equation $y = ax + b$.

  $(a, b) \longmapsto L_{(a,b)}$ defines an injective function from $\mathbb{R}^2$ to Λ.

  Hence $\mathbb{R} \sim \mathbb{R}^2 \lesssim \Lambda$.

- For each line L in $\mathbb{R}^2$, choose one ordered triple (a_L, b_L, c_L) so that L is given by the equation $a_L x + b_L y + c_L = 0$.

  $L \longmapsto (a_L, b_L, c_L)$ defines an injective function from Λ to $\mathbb{R}^3$.

  Hence $\Lambda \lesssim \mathbb{R}^3 \sim \mathbb{R}$.

- Now $\mathbb{R} \lesssim \Lambda$ and $\Lambda \lesssim \mathbb{R}$.

  According to the Schröder-Bernstein Theorem, $\Lambda \sim \mathbb{R}$.

Remark.

With similar arguments, we deduce that the set of all planes in $\mathbb{R}^3$, the set of all circles in $\mathbb{R}^2$, the set of all spheres in $\mathbb{R}^3$ et cetera are of cardinality equal to $\mathbb{R}$.

7. **Preparation for a proof of the Schöder-Bernstein Theorem.**

Recall:

(a) **Definition. (Generalized union and generalized intersection.)**

Let M be a set, and $\{S_n\}_{n=0}^{\infty}$ be an infinite sequence of subsets of the set M.

i. *The* **(generalized) intersection of the infinite sequence of subsets** $\{S_n\}_{n=0}^{\infty}$ **of the set** M *is defined to be the set* $\{x \in M : x \in S_n$ *for any* $n \in \mathbb{N}\}$. *It is denoted by* $\bigcap_{n=0}^{\infty} S_n$.

ii. *The* **(generalized) union of the infinite sequence of subsets** $\{S_n\}_{n=0}^{\infty}$ **of the set** M *is defined to be the set* $\{x \in M : x \in S_n$ *for some* $n \in \mathbb{N}\}$. *It is denoted by* $\bigcup_{n=0}^{\infty} S_n$.

(b) **Theorem (IV). ('Glueing Lemma'.)**

Let A, B be sets. Let $\{C_n\}_{n=0}^{\infty}$, $\{D_n\}_{n=0}^{\infty}$ be infinite sequences of subsets of A, B respectively. Let $\{H_n\}_{n=0}^{\infty}$ be an infinite seuqence of subsets of $A \times B$. Suppose $\{(C_n, D_n, H_n)\}_{n=0}^{\infty}$ is an infinite sequence of bijective functions. Suppose that for any $j, k \in \mathbb{N}$, if $j \neq k$ then $C_j \cap C_k = \emptyset$ and $D_j \cap D_k = \emptyset$. Then $\left(\bigcup_{n=0}^{\infty} C_n, \bigcup_{n=0}^{\infty} D_n, \bigcup_{n=0}^{\infty} H_n\right)$ is a bijective function.

8. **Outline of an argument for the Schröder-Bernstein Theorem.**

Let A, B be sets. Suppose $A \lesssim B$ and $B \lesssim A$.

Since $A \lesssim B$, there is some injective function from A to B, say, $f : A \longrightarrow B$ with graph F.

Since $B \lesssim A$, there is some injective function from B to A, say, $g : B \longrightarrow A$ with graph G.

When one of f, g is surjective as well, it will be a bijective function as well. Then we will have $A \sim B$ immediately.

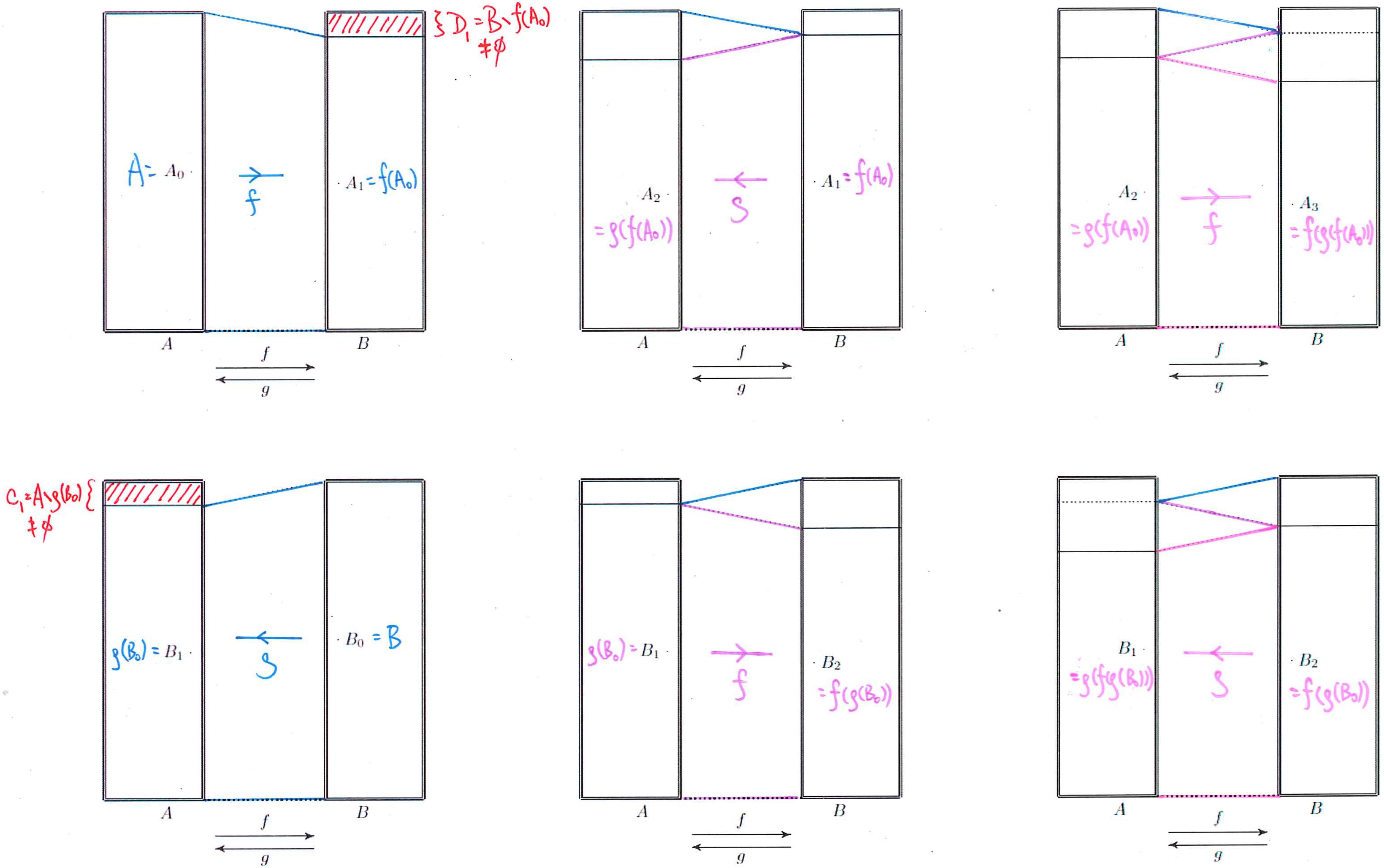
From now on, we assume that neither of f, g is surjective.

We are going to construct a bijective function from A to B out of f, g.

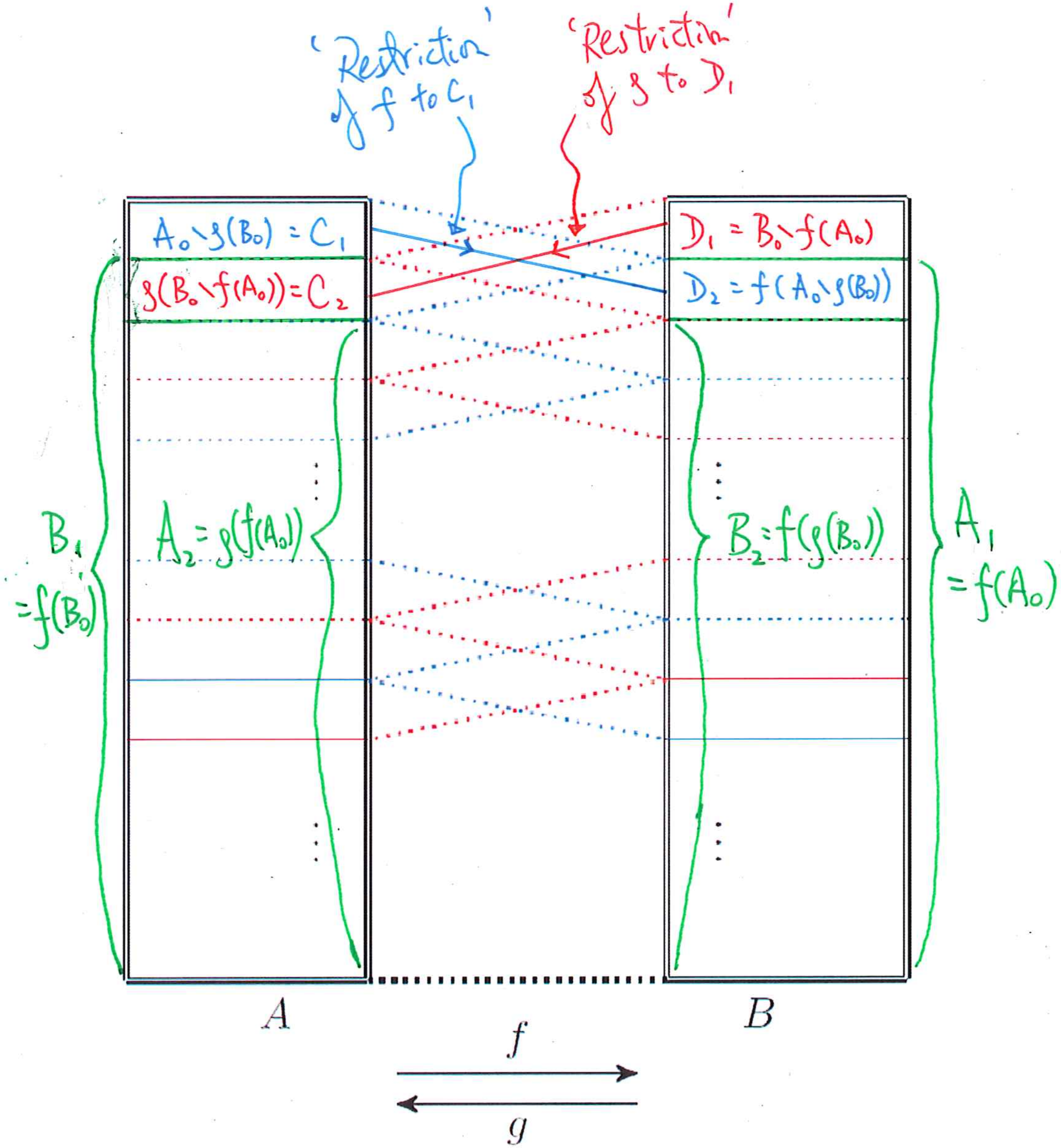
[*Idea.* Make use of the non-empty sets $B \backslash f(A)$, $A \backslash g(B)$ and the injective functions f, g to 'break up' A, B respectively into many many pieces.

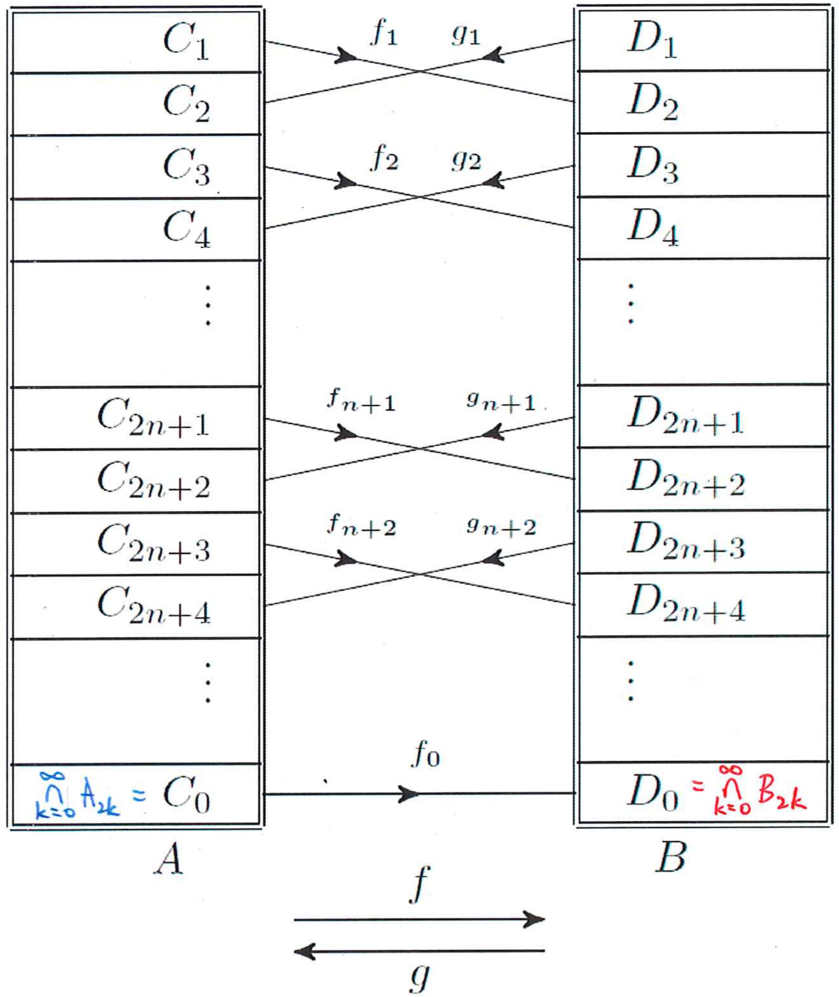
'Arrange' the 'pieces' 'on the two sides' into many many pairs appropriately, with one bijective function defined by f or g as appropriate 'joining' as its domain and range the two 'pieces' in each pair.

'Glue up' the many many bijective functions together to obtain a bijective function from A to B.]



A = A0
f
A1 = f(A0)
D1 = B \ f(A0) ≠ ∅
A
B
f
g
A2 = g(f(A0))
g
A1 = f(A0)
A2 = g(f(A0))
f
A3 = f(g(f(A0)))
C1 = A \ g(B0) ≠ ∅
g(B0) = B1
g
B0 = B
g(B0) = B1
f
B2 = f(g(B0))
B1 = g(f(g(B0)))
g
B2 = f(g(B0))





$C_1 \sim D_2$ 'due to f'

$D_1 \sim C_2$ 'due to g'