MATH1050 Sets of not necessarily the same size

1. **Definition.**
   Let A, B be sets.
   (a) *A is said to be* **of cardinality less than or equal to** *B if there is an injective function from A to B. We write $A \lesssim B$.*
   We may also write $B \gtrsim A$ and say B is of cardinality greater than or equal to A.
   (b) *A is said to be* **of cardinality less than** *B if (there is an injective function from A to B and there is no bijective function from A to B). We write $A < B$.*
   We may also write $B > A$ and say B is of cardinality greater than A.

   Remark on further notations.
   - We write $A \not\lesssim B$, or equivalently $B \not\gtrsim A$ exactly when it is not true that $A \lesssim B$,
   - We write $A \not< B$, or equivalently $B \not> A$ exactly when it is not true that $A < B$.

2. **Theorem (IX). (Basic properties of $\lesssim$.)**
   (0) *Let A, B be sets. Suppose $A \sim B$. Then $A \lesssim B$.*
   (1) *Let A, B be sets. $A < B$ iff ($A \lesssim B$ and $A \not\sim B$).*
   (2) *Let A, B be sets. Suppose $A \subset B$. Then $A \lesssim B$.*
   (3) *Let A be a set. $\emptyset \lesssim A$. If $A \lesssim \emptyset$ then $A = \emptyset$.*
   (4) *Let A be a set. $A \lesssim A$.*
   (5) *Let A, B, C be sets. Suppose $A \lesssim B$ and $B \lesssim C$. Then $A \lesssim C$.*

3. **Simple example (1).**
   $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.
   Then $\mathbb{N} \lesssim \mathbb{Z} \lesssim \mathbb{Q} \lesssim \mathbb{R} \lesssim \mathbb{C}$.

4. **Simple example (2).**
   $\mathbb{Q} \lesssim \mathbb{Z}^2$.

   Remark. Recall that $\mathbb{Z}^2 \sim \mathbb{N}^2 \sim \mathbb{N}$. Then $\mathbb{Q} \lesssim \mathbb{N}$.

   Justification for $\mathbb{Q} \lesssim \mathbb{Z}^2$:
   - We take the statement ($\sharp$) for granted:
     ($\sharp$) *For any $r \in \mathbb{Q} \backslash \{0\}$, there exist some unique $p_r, q_r \in \mathbb{Z}$ such that $\gcd(p_r, q_r) = 1$ and $q_r > 0$ and $r = \dfrac{p_r}{q_r}$.*
   - Define the function $f : \mathbb{Q} \longrightarrow \mathbb{Z}^2$ by
     $$f(r) = \begin{cases} (p_r, q_r) & \text{if} \quad r \in \mathbb{Q} \backslash \{0\} \\ (0, 1) & \text{if} \quad r = 0. \end{cases}$$
     f is injective. (Exercise.)
     It follows that $\mathbb{Q} \lesssim \mathbb{Z}^2$.
   - Justification for the statement ($\sharp$)? Exercise. (Refer to the Handout *Basic results on divisibility* and the Handout *Euclidean Algorithm.*)

5. **Theorem (X). (Further basic properties of $\lesssim$.)**
   (1) *Let $f : A \longrightarrow B$ be a function. The following statements hold:*
   (1a) *If f is injective then $f(A) \sim A$.*
   (1b) *$f(A) \lesssim A$.*
   (2) *Let A, B be non-empty sets. $A \lesssim B$ iff (there is a surjective function from B to A.)*
   (3) *Let A, B, C, D be sets. Suppose $A \lesssim C$ and $B \lesssim D$. Then $A \times B \lesssim C \times D$.*

(4) *Let A, B be sets. Suppose $A \lesssim B$. Then $\mathfrak{P}(A) \lesssim \mathfrak{P}(B)$.*

(5) *Let A, B, C, D be non-empty sets. Suppose $A \lesssim C$ and $B \lesssim D$. Then $\mathsf{Map}(A, B) \lesssim \mathsf{Map}(C, D)$.*

6. **Two seemingly obvious but non-trivial results about $\lesssim$.**

Question.

Consider each of the statements below. Is it true? Or is it false?

(1) '*Let A, B be sets. Suppose $A \lesssim B$ and $B \lesssim A$. Then $A = B$.* '

(2) '*Let A, B be sets. $A \lesssim B$ or $B \lesssim A$. (Exactly one of '$A < B$', '$A = B$', '$A > B$' holds.)* '

Answer.

Statement (1) is false; counter-example? Statement (2) is true; proof?

Remark on the question. Why are we bothered with such a question? Recall that these statements are true:

- '*Let A be a set. $A \lesssim A$.*'
- '*Let A, B, C be sets. Suppose $A \lesssim B$ and $B \lesssim C$. Then $A \lesssim C$.*'

It is natural to ask whether

$$\lesssim \text{ 'behaves like' a partial ordering or total ordering,}$$

as the symbol '$\lesssim$' suggests.

Statement (1) is false in the sense 'because' it takes too much for a 'set equality' to hold. 'Relaxing' the 'conclusion part' in Statement (1), we do obtain an important true statement.

Theorem (XI). (Schröder-Bernstein Theorem.)

Let A, B be sets. Suppose $A \lesssim B$ and $B \lesssim A$. Then $A \sim B$.

Statement (2) is true and is highly non-trivial; it is a consequence of the Axiom of Choice, under the other 'standard assumptions' of set theory.

Theorem (XII). (Law of Trichotomy.)

Let A, B be sets. $A \lesssim B$ or $B \lesssim A$. (Exactly one of '$A < B$', '$A \sim B$', '$A > B$' holds.)

7. **Axiom of Choice**.

What is this so called Axiom of Choice? There are many (logially equivalent) formulations; below are two of them:

- **(AC1)** *Let I, M be non-empty sets, and $\Phi : I \longrightarrow \mathfrak{P}(M)$ be a function. Suppose $\Phi(\alpha) \neq \emptyset$ for any $\alpha \in I$. Then there exists a function $\varphi : I \longrightarrow M$ such that $\varphi(\alpha) \in \Phi(\alpha)$ for any $\alpha \in I$.*
- **(AC2)** *For any non-empty set A, there exists some function $\psi : \mathfrak{P}(A) \backslash \{\emptyset\} \longrightarrow A$ such that $\psi(S) \in S$ for any $S \in \mathfrak{P}(A) \backslash \{\emptyset\}$.*
- **(AC3)** *The cartesian product of any non-empty family of non-empty sets is non-empty.*

Remark.

In the context of the statement (AC1), it is the function φ through which we *choose to assign* each $\alpha \in I$ to the element $\varphi(\alpha)$ of the subset $\Phi(\alpha)$ of M. In the context of the statement (AC2), the function ψ is called a **choice function**: it is through ψ that we *choose* for each non-empty subset of A an element of the subset concerned.

Why such functions exist in the first place is the problem: intuition suggests they *must* exist, but intuition cannot take the place of reason.

The statement (AC3) is what is usually presented as the statement of Axiom of Choice in standard *axiomatic set theory* textbooks.

Further remark. Refer to Theorem (X).

(a) We need the Axiom of Choice (the statement (AC1)) in the proof of (1b).

(b) We also need the Axiom of Choice (the statement (AC1)) in the argument for '$\Longleftarrow$-part' of (2).