1. **Example (A). ('Congruence modulo n'.)**

   Let $n \in \mathbb{N}$. This will be kept fixed throughout the discussion below.

   We introduce the notion of congruence modulo n.

   Definition.

   Let $x, y \in \mathbb{Z}$. x is said to be **congruent to y modulo n** *if $x - y$ is divisible by n. We write $x \equiv y (\text{mod } n)$.*

   Lemma (A1).

   The following statements hold:

   (ρ): *For any $x \in \mathbb{Z}$, $x \equiv x(\text{mod } n)$.*

   (σ): *For any $x, y \in \mathbb{Z}$, if $x \equiv y(\text{mod } n)$ then $y \equiv x(\text{mod } n)$.*

   (τ): *For any $x, y, z \in \mathbb{Z}$, if $x \equiv y(\text{mod } n)$ and $y \equiv z(\text{mod } n)$ then $x \equiv z(\text{mod } n)$.*

   From now on assume $n \geq 2$.

   Define $E_n = \{(x, y) \mid x, y \in \mathbb{Z} \text{ and } x \equiv y(\text{mod } n)\}$, and $R_n = (\mathbb{Z}, \mathbb{Z}, E_n)$.

   According to Lemma ($\star$), R_n is an equivalence relation in $\mathbb{Z}$. (Proof? Exercise.)

   Through R_n, we disregard the distinction between two (different) numbers exactly when their difference is divisible by n. But the latter happens exactly when the two numbers concerned have the same remainder upon division by n.

2. **Example (B). (Parallelism in the 'infinite plane'.)**

   Recall how parallelism in the 'infinite plane' is understood in school geometry:

   - *Given any two distinct lines in the plane, one is parallel to the other exactly when they have no intersection.*

   (For the ancient Greeks' definition, refer to Definition 23, Book I of Euclid's *Elements*. It reads: '*Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.*')

   These properties of parallelism is taken for granted:

   (1) *Given any two (distinct) lines in the plane, if one of them is parallel to the other then the latter is parallel to the former.*
   (This is implicit in the definition.)

   (2) *Given any three (distinct) lines in the plane, if the first is parallel to the second and the second is parallel to the third then the first is parallel to the third.*
   (See Proposition 30, Book I of Euclid's *Elements*.)

   Here we extend the notion of parallelism by a little bit so that the extended notion defines an equivalence relation in the set of all lines in the 'infinite plane'. This 'little bit of extension of definition' is that we declare that every line is parallel to itself.

   Definition.

   Let ℓ, m be lines in $\mathbb{R}^2$ (regarded as subsets of $\mathbb{R}^2$).

   ℓ is said to be **parallel** *to m if ($\ell = m$ or $\ell \cap m = \emptyset$).*

   Let Λ be the set of all lines in $\mathbb{R}^2$. Define $P = \{(\ell, m) \mid \ell, m \in \Lambda \text{ and } \ell \text{ is parallel to } m\}$.

   (Λ, Λ, P) is an equivalence relation: reflexivity is guaranteed by the 'little bit of extension of definition' made by us here, whereas symmetry and transitivity are essentially guaranteed by the properties of parallelism known to us in school geometry, as listed earlier.

   Through this equivalence relation, we disregard the distinction between two distinct lines exactly when the lines concerned are parallel to each other.

   The above idea can be generalized to parallelism for lines in $\mathbb{R}^3$ and parallelism for planes in $\mathbb{R}^3$.

   Definitions.

   (a) *Let ℓ, m be lines in $\mathbb{R}^3$. ℓ is said to be* **parallel** *to m if ((ℓ is identical to m) or (ℓ, m lie on the same plane and ℓ, m have no intersection in $\mathbb{R}^3$)).*

(b) *Let π, ρ be planes in $\mathbb{R}^3$. π is said to be* **parallel** *to ρ if ((π is identical to ρ) or (π, ρ has no intersection in $\mathbb{R}^3$)).*

Let $\Lambda_{1,3}$ be the set of all lines in $\mathbb{R}^3$, and $E_{1,3} = \{(\ell, m) \mid \ell, m \in \Lambda_{1,3} \text{ and } \ell \text{ is parallel to } m\}$. $(\Lambda_{1,3}, \Lambda_{1,3}, E_{1,3}))$ is an equivalence relation.

Let $\Lambda_{2,3}$ be the set of all planes in $\mathbb{R}^3$, and $E_{2,3} = \{(\pi, \rho) \mid \pi, \rho \in \Lambda_{2,3} \text{ and } \pi \text{ is parallel to } \rho\}$. $(\Lambda_{2,3}, \Lambda_{2,3}, E_{2,3}))$ is an equivalence relation.

The same idea can be further generalized to higher dimensional geometry: parallelism for k-dimensional hyperplanes in the n-dimensional space $\mathbb{R}^n$.

3. **Example (C). (Congruence in Euclidean geometry.)**

In school maths we learnt the notion of 'congruence for geometric figures in the plane', with special emphasis on 'congruent triangles'.

The typical 'textbook definition' for the notion of congruence might have read:

- *Two plane figures are congruent exactly when they are of the same shape and of the same size.*

Then came results like 'SAS', 'SSS', 'ASA', 'AAS', which give various 'sufficient conditions' for pairs of triangles to be congruent. Probably the symbol '$\cong$' was introduced in the context. This symbol would obey certain rules:

(ρ): $\triangle ABC \cong \triangle ABC$.

(σ): *Suppose* $\triangle ABC \cong \triangle DEF$. *Then* $\triangle DEF \cong \triangle ABC$.

(τ): *Suppose* $\triangle ABC \cong \triangle DEF$ *and* $\triangle DEF \cong \triangle JKL$. *Then* $\triangle ABC \cong \triangle JKL$.

These rules suggest that some kind of equivalence relations is lurking behind the notion of 'congruence for geometric figures in the plane'.

Let $n \in \mathbb{N} \backslash \{0\}$. This will be kept fixed throughout the discussion below.

Definition.

Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a bijective function.

φ is called an **isometry** *in $\mathbb{R}^n$ if the statement* (DP) *holds:*

(DP) For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\|\varphi(\mathbf{x}) - \varphi(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$.

Remark. We can in fact drop the assumption on bijectivity in the definition of the notion of isometry. This is due to the validity of the statement below:

Let $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function. Suppose that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\|\psi(\mathbf{x}) - \psi(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$. Then there exist some $(n \times n)$-orthogonal matrix A with real entries and some $\mathbf{b} \in \mathbb{R}^n$ such that for any $\mathbf{x} \in \mathbb{R}^n$, $\psi(x) = A\mathbf{x} + \mathbf{b}$.

Such a function ψ is bijective.

Definition.

Let S, T be subsets of $\mathbb{R}^n$.

(a) *Let φ be an isometry in $\mathbb{R}^n$. The set S is said to be* **congruent to** *the set T* **under the isometry** *φ if $T = \varphi(S)$. We write $S \cong_\varphi T$.*

(b) *The set S is said to be* **congruent to** *the set T if there exists some isometry ψ in $\mathbb{R}^n$ such that $T = \psi(S)$. When we do not emphasize which isometry ψ is, we agree to write $S \cong T$.*

Lemma (C1).

The following statements hold:

(ρ): *For any $S \in \mathfrak{P}(\mathbb{R}^n)$, $S \cong S$.*

(σ): *For any $S, T \in \mathfrak{P}(\mathbb{R}^n)$, if $S \cong T$ then $T \cong S$.*

(τ): *For any $S, T, U \in \mathfrak{P}(\mathbb{R}^n)$, if $S \cong T$ and $T \cong U$ then $S \cong U$.*

We define the **Euclidean congruence in** $\mathbb{R}^n$ to be the relation in $\mathfrak{P}(\mathbb{R}^n)$ with graph

$$E_{\cong,n} = \{(S,T) \mid S,T \in \mathfrak{P}(\mathbb{R}^n) \text{ and } S \cong T\}.$$

The Euclidean congruence in $\mathbb{R}^n$ is an equivalence relation in the set $\mathfrak{P}(\mathbb{R}^n)$.

Through this equivalence relation, we disregard the distinction between two distinct subsets in $\mathbb{R}^n$ exactly when they are of the same shape and the same size (so that the image set of one subset under an appropriate isometry 'fits perfectly' onto the other subset).

Now 'congruence of triangles in the plane' in school geometry can be seen as the Euclidean congruence in $\mathbb{R}^2$ 'restricted' to some subset of $\mathfrak{P}(\mathbb{R}^2)$, namely, the set of all triangles in $\mathbb{R}^2$.

Definition.

Let T be a subset of $\mathbb{R}^2$. T is said to be a **triangle** *in $\mathbb{R}^2$ if T is the union of three non-collinear points in $\mathbb{R}^2$ and the three line segments joining the respective pairs of in those three points.*

Remark. Just as congruence for triangles in the Euclidean plane is a special case of congruence in the Euclidean plane/space/..., similarity for triangles in the Euclidean plane is a special case of **'similarity in the Euclidean plane/space/...'**. (As an exercise, find out what the appropriate formulation for the latter is.)

4. **Example (D). (Row-equivalence for matrices.)**

Let $p, q \in \mathbb{N}\backslash\{0\}$. They will be kept fixed throughout the discussion below.

Definition.

Let C, D be $(p \times q)$-matrices with real entries. We say C is **row-equivalent** *to D if there is a finite sequence of row operations starting from C and ending at D.*

Row-equivalence defines an equivalence relation in the set of all $(p\times q)$-matrices with real entries, by virtue of the validity of Theorem (D1).

Theorem (D1).

The statements (ρ), (σ), (τ) holds:

(ρ): *Suppose A is a $(p \times q)$-matrix with real entries. Then A is row-equivalent to A.*

(σ): *Let A, B be $(p \times q)$-matrices with real entries. Suppose A is row-equivalent to B. Then B is row-equivalent to A.*

(τ): *Let A, B, C be $(p \times q)$-matrices with real entries. Suppose A is row-equivalent to B, and B is row-equivalent to C. Then A is row-equivalent to C.*

Define $E = \{(A, B) \mid A, B \in \mathsf{Mat}_{p\times q}(\mathbb{R}) \text{ and } A \text{ is row-equivalent to } B\}$, and $R = (\mathsf{Mat}_{p\times q}(\mathbb{R}), \mathsf{Mat}_{p\times q}(\mathbb{R}), E)$.

According to Theorem (D1), R is an equivalence relation in $\mathsf{Mat}_{p\times q}(\mathbb{R})$.

Through this equivalence relation, we disregard the distinction between two distinct $(p \times q)$-matrices with real entries exactly when they are row-equivalent to each other.

5. **Example (E). (Sets of equal cardinality.)**

Recall the definition for the notion of equipotency:

Let S, T be sets. We say that S is **of cardinality equal to** *T, and write $S{\sim}T$, if there is a bijective function from S to T.*

Let M be a set. This is kept fixed throughout the discussion below.

Theorem (E1).

The statements (ρ), (σ), (τ) hold:

(ρ): *Suppose $A \in \mathfrak{P}(M)$. Then $A{\sim}A$.*

(σ): *Let $A, B \in \mathfrak{P}(M)$. Suppose $A{\sim}B$. Then $B{\sim}A$.*

(τ): *Let $A, B, C \in \mathfrak{P}(M)$. Suppose $A{\sim}B$ and $B{\sim}C$. Then $A{\sim}C$.*

Define $E_P = \{(A, B) \mid A, B \in \mathfrak{P}(M) \text{ and } A \sim B\}$, and $R_P = (\mathfrak{P}(M), \mathfrak{P}(M), E_P)$.

According to Theorem (E1), R_P is an equivalence relation in $\mathfrak{P}(M)$.

Through the equivalence relation R_P, we disregard the distinction between two distinct subsets of M exactly when they are of equal cardinality to each other.

6. **Example (F). ('Contours' and 'level sets'.)**

   (a) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function defined by $f(x, y) = x^2 + y^2$ for any $x, y \in \mathbb{R}$. This is kept fixed throughout the discussion below.

   The statements below hold:

   (ρ): *For any* $p, q \in \mathbb{R}$, $f(p, q) = f(p, q)$.

   (σ): *For any* $p, q, s, t \in \mathbb{R}$, *if* $f(p, q) = f(s, t)$ *then* $f(s, t) = f(p, q)$.

   (τ): *For any* $p, q, s, t, u, v \in \mathbb{R}$, *if* $f(p, q) = f(s, t)$ *and* $f(s, t) = f(u, v)$ *then* $f(p, q) = f(u, v)$.

   Define $E_f = \{((p, q), (s, t)) \mid p, q, s, t \in \mathbb{R} \text{ and } f(p, q) = f(s, t)\}$, and $R_f = (\mathbb{R}^2, \mathbb{R}^2, E_f)$.

   R_f is an equivalence relation in $\mathbb{R}^2$. It is (naturally) induced by the function f.

   Through the equivalence relation R_f, we disregard the distinction between two distinct points in $\mathbb{R}^2$ exactly when they belong to the same level set of f.

   Each such (non-empty) level set of f is a circle with centre at the origin.

   Remark. The equivalence relation R_f can be understood through ($\star_f$), in terms of solving equations:

   ($\star_f$) *For any* $p, q, s, t \in \mathbb{R}$, $((p, q), (s, t)) \in E_f$ *iff there exists some* $c \in \mathbb{R}$ *such that* '$(x, y) = (p, q)$', '$(x, y) = (s, t)$' *are solutions of the equation* $x^2 + y^2 = c$ *with unknown* x, y *in* $\mathbb{R}$.

   (b) Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function defined by $g(x, y) = x^2 - y^2$ for any $x, y \in \mathbb{R}$. This is kept fixed throughout the discussion below.

   The statements below hold:

   (ρ): *For any* $p, q \in \mathbb{R}$, $g(p, q) = g(p, q)$.

   (σ): *For any* $p, q, s, t \in \mathbb{R}$, *if* $g(p, q) = g(s, t)$ *then* $g(s, t) = g(p, q)$.

   (τ): *For any* $p, q, s, t, u, v \in \mathbb{R}$, *if* $g(p, q) = g(s, t)$ *and* $g(s, t) = g(u, v)$ *then* $g(p, q) = g(u, v)$.

   Define $E_g = \{((p, q), (s, t)) \mid p, q, s, t \in \mathbb{R} \text{ and } g(p, q) = g(s, t)\}$, and $R_g = (\mathbb{R}^2, \mathbb{R}^2, E_g)$.

   R_g is an equivalence relation in $\mathbb{R}^2$. It is (naturally) induced by the function g.

   Through the equivalence relation R_g, we disregard the distinction between two distinct points in $\mathbb{R}^2$ exactly when they belong to the same level set of g.

   Each such (non-empty) level set of g is a hyperbola with centre at the origin and with asymptotes '$y = x$', '$y = -x$'.

   Remark. The equivalence relation R_g can be understood through ($\star_g$), in terms of solving equations:

   ($\star_g$) *For any* $p, q, s, t \in \mathbb{R}$, $((p, q), (s, t)) \in E_g$ *iff there exists some* $c \in \mathbb{R}$ *such that* '$(x, y) = (p, q)$', '$(x, y) = (s, t)$' *are solutions of the equation* $x^2 - y^2 = c$ *with unknown* x, y *in* $\mathbb{R}$.

7. **Example (G). (Solutions of systems of linear equations with a common matrix of coefficients.)**

   Let A be an $(m \times n)$-matrix with real entries. This matrix A is fixed throughout the discussion.

   The statements below hold:

   (ρ): *For any* $\mathbf{u} \in \mathbb{R}^n$, $A\mathbf{u} = A\mathbf{u}$.

   (σ): *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, *if* $A\mathbf{u} = A\mathbf{v}$ *then* $A\mathbf{v} = A\mathbf{u}$.

   (τ): *For any* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, *if* $A\mathbf{u} = A\mathbf{v}$ *and* $A\mathbf{v} = A\mathbf{w}$ *then* $A\mathbf{u} = A\mathbf{w}$.

   Define the relation $S_A = (\mathbb{R}^n, \mathbb{R}^n, E_A)$ by $E_A = \{(\mathbf{u}, \mathbf{v}) \mid \mathbf{u}, \mathbf{v} \in \mathbb{R}^n \text{ and } A\mathbf{u} = A\mathbf{v}\}$.

   S_A is an equivalence relation in $\mathbb{R}^n$.

   The equivalence relation S_A can be understood through ($\star_A$), in terms of solving equations:

   ($\star_A$) $(\mathbf{u}, \mathbf{v}) \in E_A$ *iff there exists some* $\mathbf{b} \in \mathbb{R}^m$ *such that* $\mathbf{u}, \mathbf{v}$ *belong to the solution set of the equation* $A\mathbf{x} = \mathbf{b}$ *with unknown* $\mathbf{x}$ *in* $\mathbb{R}^n$.

Therefore, through the equivalence relation S_A, we disregard the distinction between two distinct vectors in $\mathbb{R}^n$ exactly when both are solutions to the equation with 'coefficient matrix' A and with the same 'vector of constant'.

Remark. S_A can be seen to be the equivalence relation (naturally) induced by a function from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Define the function $L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by $L_A(\mathbf{x}) = A\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$.

(L_A is called the **linear transformation defined by matrix multiplication from the left by** A.)

By definition, for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $(\mathbf{u}, \mathbf{v}) \in E_A$ iff $L_A(\mathbf{u}) = L_A(\mathbf{v})$.

Therefore, through the equivalence relation S_A, we disregard the distinction between two distinct vectors in $\mathbb{R}^n$ exactly when they belong to the same level set of L_A.

8. **Example (H). (Primitives of continuous functions.)**

Let I be an open interval in $\mathbb{R}$. This is kept fixed throughout the discussion below.

Denote by $C^1(I)$ the set of all real-valued functions with domain I which is continuously differentiable on I.

Differentiation defines an equivalence relation in $C^1(I)$, by virtue of the validity of Theorem (H1).

Theorem (H1).

The statements (ρ), (σ), (τ) *hold:*

- (ρ): *Suppose* $f \in C^1(I)$. *Then* $f' = f'$ *as functions.*
- (σ): *Let* $f, g \in C^1(I)$. *Suppose* $f' = g'$ *as functions. Then* $g' = f'$ *as functions.*
- (τ): *Let* $f, g, h \in C^1(I)$. *Suppose* $f' = g'$ *as functions and* $g' = h'$ *as functions. Then* $f' = h'$ *as functions.*

Define $E_D = \{(f, g) \mid f, g \in C^1(I) \text{ and } f' = g'\}$, and $R_D = (C^1(I), C^1(I), E_D)$.

R_D is an equivalence relation in $C^1(I)$.

Through the equivalence relation R_D, we disregard the distinction between two distinct continuously differentiable functions on I exactly when they are primitives of the same continuous function on I.