1. **Definition.**

Let A, B be sets. A is said to be **of cardinality equal to** *B if there is a bijective function from A to B. We write $A \sim B$.*

Remark on notation. Where A is not of cardinality equal to B, we write $A \not\sim B$.

2. **Theorem (I). (Properties of $\sim$.)**

(1) *Let A be a set. $A \sim \emptyset$ iff $A = \emptyset$.*

(2) *Let x, y be objects. $\{x\} \sim \{y\}$.*

(3) *Let A, B, C be sets.*

The following statements hold:

(3a) *$A \sim A$.*

(3b) *Suppose $A \sim B$. Then $B \sim A$.*

(3c) *Suppose $A \sim B$ and $B \sim C$. Then $A \sim C$.*

(4) *Let A, B, C, D be sets.*

The following statements hold:

(4a) *Suppose $A \sim C$ and $B \sim D$. Then $A \times B \sim C \times D$.*

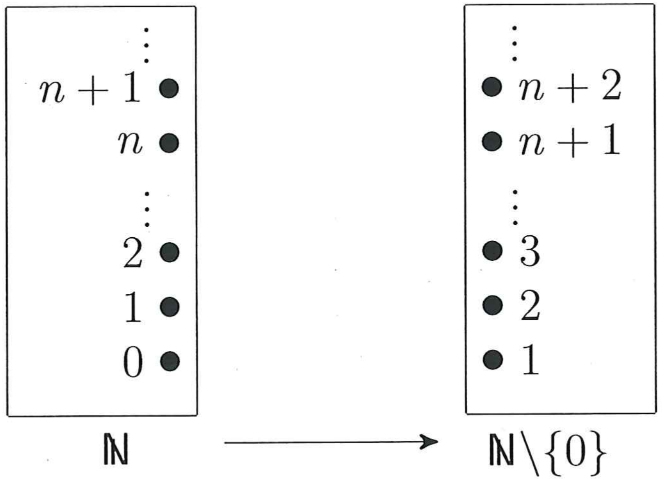
(4b) *Suppose $A \sim C$. Then $\mathfrak{P}(A) \sim \mathfrak{P}(C)$.*

(4c) *Suppose $A \sim C$ and $B \sim D$. Then $\mathsf{Map}(A, B) \sim \mathsf{Map}(C, D)$.*

3. **Example (α).**

$\mathbb{N} \sim \mathbb{N} \backslash \{0\}$.

(a) *Idea.*



'Blobs-and-arrows' diagram for some bijective function $f : \mathbb{N} \longrightarrow \mathbb{N}\backslash\{0\}$?

Graph of f?

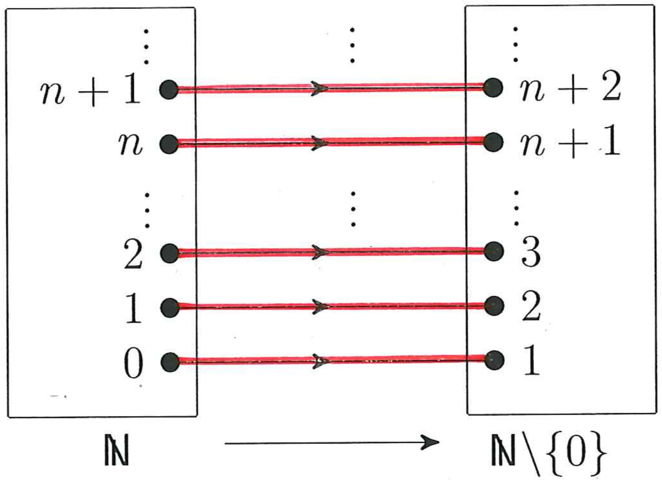
Formula of definition of f?

(b) *Formal argument.*

Example (α).

$\mathbb{N} \sim \mathbb{N} \backslash \{0\}$.

(a) *Idea.*



'Blobs-and-arrows' diagram for some bijective function $f : \mathbb{N} \longrightarrow \mathbb{N} \backslash \{0\}$?

Graph of f?

$\{ (x, x+1) \mid x \in \mathbb{N} \}$

Formula of definition of f?

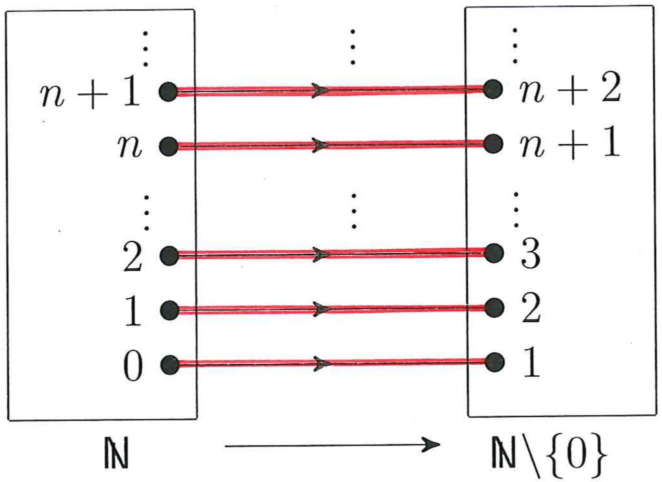
$f(x) = x+1$ for any $x \in \mathbb{N}$.

(b) *Formal argument.*

Example (α).

$\mathbb{N} \sim \mathbb{N} \backslash \{0\}$.

(a) *Idea.*



'Blobs-and-arrows' diagram for some bijective function $f : \mathbb{N} \longrightarrow \mathbb{N} \backslash \{0\}$?

Graph of f?

$\{(x, x+1) \mid x \in \mathbb{N}\}$.

Formula of definition of f?

$f(x) = x + 1$ for any $x \in \mathbb{N}$.

$$F = \left\{ (x,y) \;\middle|\; \begin{array}{l} x, y \in \mathbb{N} \text{ and} \\ \text{there exists some } t \in \mathbb{N} \\ \text{such that } x = t \text{ and } y = t+1. \end{array} \right\}$$

(b) *Formal argument.*

Let $F = \{(x, x+1) \mid x \in \mathbb{N}\}$.

Note that $F \subset \mathbb{N} \times (\mathbb{N} \backslash \{0\})$.

Define $f = (\mathbb{N}, \mathbb{N} \backslash \{0\}, F)$.

f is a relation from $\mathbb{N}$ to $\mathbb{N} \backslash \{0\}$.

Now verify that f is a bijective function.

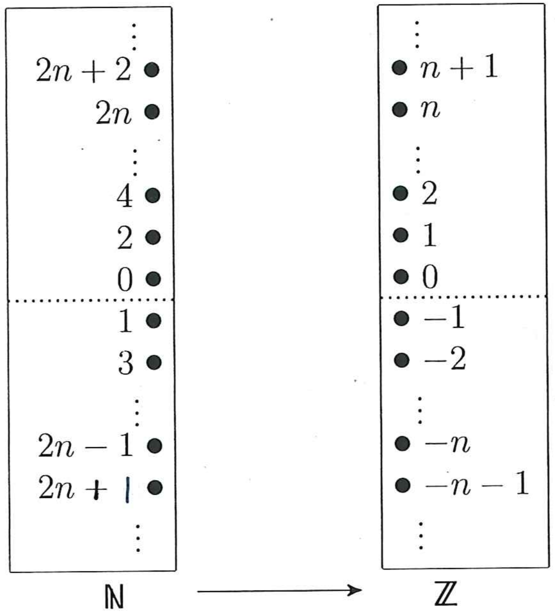
What to check?

- Condition (E).
- Condition (U).
- Condition (S).
- Condition (I).

4. **Example (β).**

$\mathbb{N} \sim \mathbb{Z}$.

(a) *Idea.*



'Blobs-and-arrows' diagram for some bijective function $f : \mathbb{N} \longrightarrow \mathbb{Z}$?

Formula of definition of f?

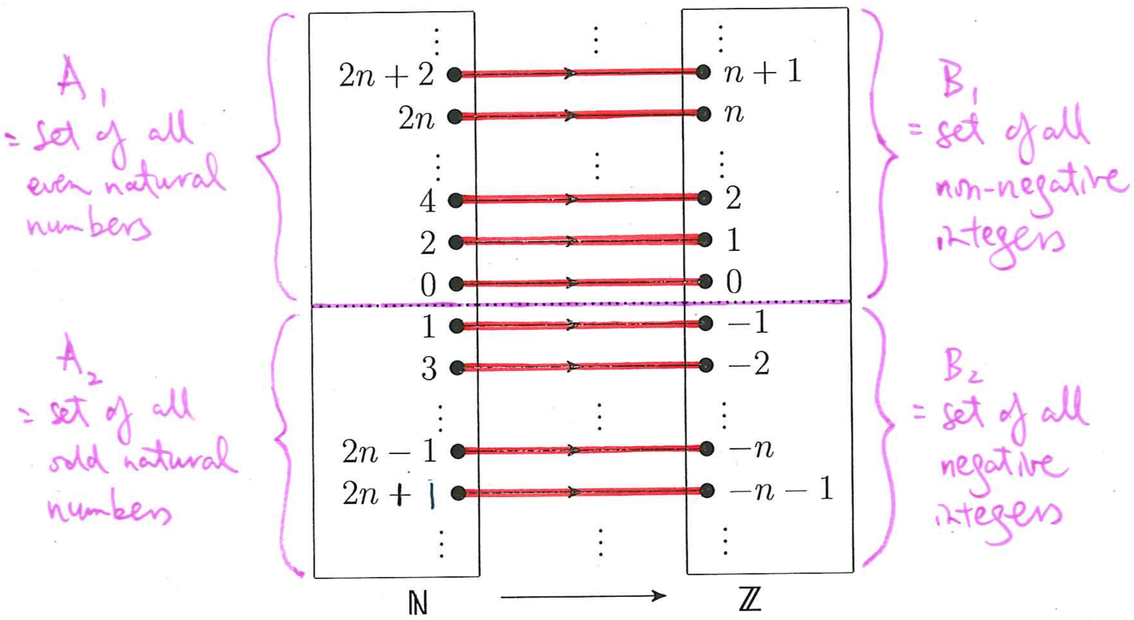
(b) *Formal argument.*

Example (β).

$\mathbb{N} \sim \mathbb{Z}$.

(a) *Idea.*

(b) *Formal argument.*



'Blobs-and-arrows' diagram for some bijective function $f : \mathbb{N} \longrightarrow \mathbb{Z}$?

Formula of definition of f?

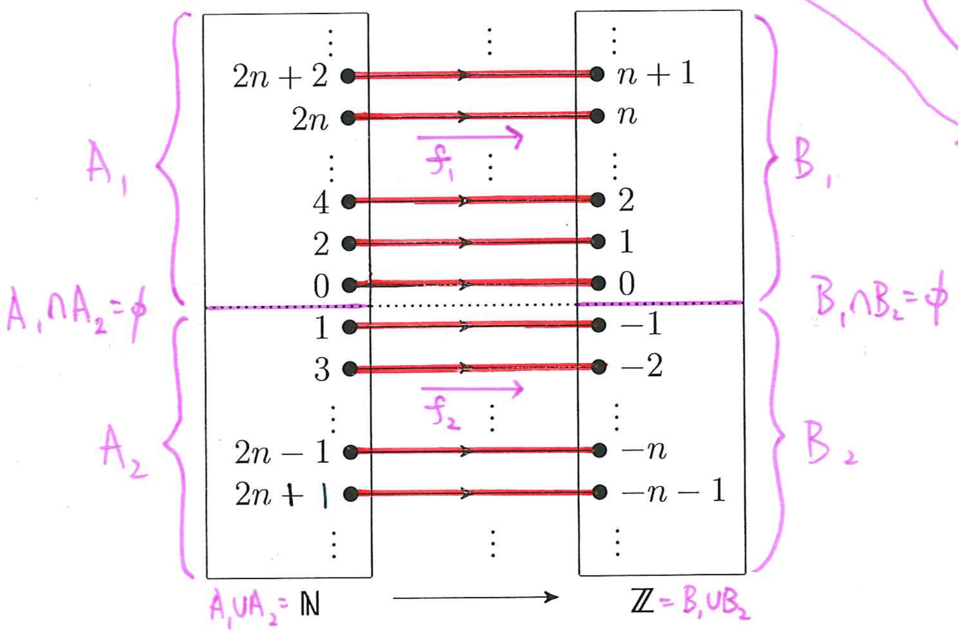
$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even.} \\ -\frac{x+1}{2} & \text{if } x \text{ is odd.} \end{cases}$$

Example (β).

$\mathbb{N} \sim \mathbb{Z}$.

$f_1 : A_1 \to B_1$ is the bijective function with graph F_1.
$f_2 : A_2 \to B_2$ is the bijective function with graph F_2.

(a) *Idea.*



'Blobs-and-arrows' diagram for some bijective function $f : \mathbb{N} \longrightarrow \mathbb{Z}$?

Formula of definition of f?

$$f(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ -(x+1)/2 & \text{if } x \text{ is odd} \end{cases}$$

(b) *Formal argument.*

Let

$F_1 = \{(2x, x) \mid x \in \mathbb{N}\}$,

$F_2 = \{(2x-1, -x) \mid x \in \mathbb{N}\backslash\{0\}\}$,

and $F = F_1 \cup F_2$.

Note that $F \subset \mathbb{N} \times \mathbb{Z}$.

Define $f = (\mathbb{N}, \mathbb{Z}, F)$.

f is a relation from $\mathbb{N}$ to $\mathbb{Z}$.

Now verify that f is a bijective function.

What is f, really?

It is the bijective function obtained by 'glueing together' the bijective functions $f_1 = (A_1, B_1, F_1)$, $f_2 = (A_2, B_2, F_2)$.

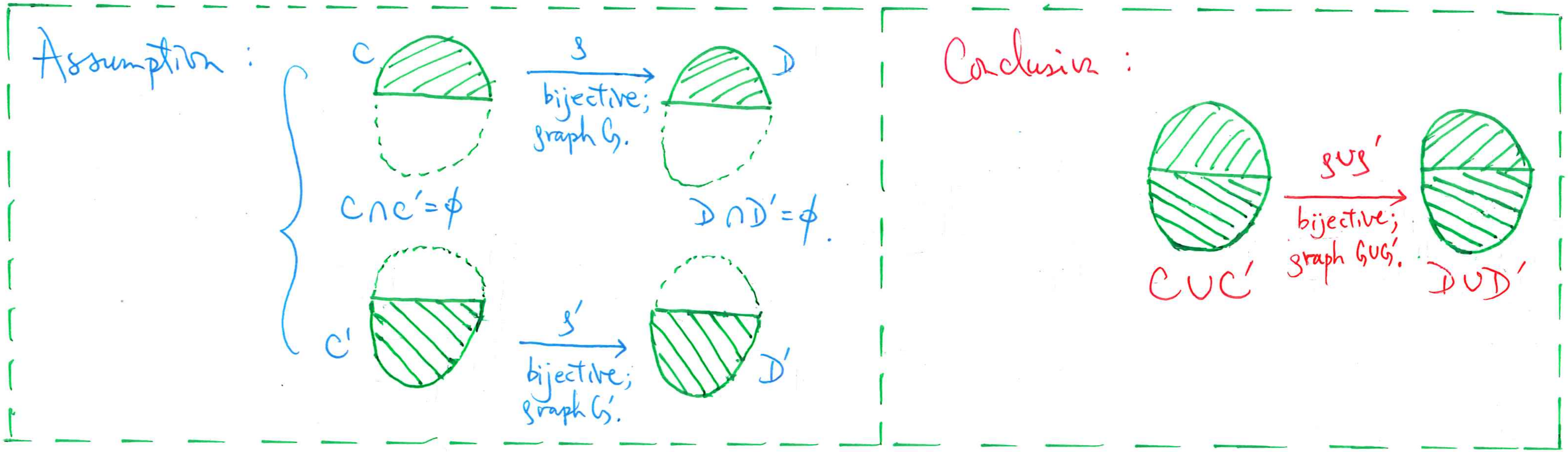
5. **'Glueing Lemma'.**

Theorem (II). ('Baby version' of 'Glueing Lemma').

Let C, C', D, D' *be sets, and* $g = (C, D, G)$, $g' = (C', D', G')$ *be bijective functions.*

Suppose $C \cap C' = \emptyset$ *and* $D \cap D' = \emptyset$.

Then $(C \cup C', D \cup D', G \cup G')$ *is a bijective function.*



Corollary (III).

Let C, C', D, D' *be sets. Suppose* $C \sim D$ *and* $C' \sim D'$.

Also suppose $C \cap C' = \emptyset$ *and* $D \cap D' = \emptyset$.

Then $C \cup C' \sim D \cup D'$.

‘Glueing Lemma’.

Theorem (II) and Corollary (III) may be extended to the situation for infinite sequences of sets and generalized unions:

Theorem (IV). (‘Glueing Lemma’.)

Let A, B be sets.

Let $\{C_n\}_{n=0}^{\infty}$, $\{D_n\}_{n=0}^{\infty}$ be infinite sequences of subsets of A, B respectively.

Let $\{G_n\}_{n=0}^{\infty}$ be an infinite seuqence of subsets of $A \times B$.

Suppose $\{(C_n, D_n, G_n)\}_{n=0}^{\infty}$ is an infinite sequence of bijective functions.

Suppose that for any $j, k \in \mathbb{N}$, if $j \neq k$ then $C_j \cap C_k = \emptyset$ and $D_j \cap D_k = \emptyset$.

Then $\left(\bigcup_{n=0}^{\infty} C_n, \bigcup_{n=0}^{\infty} D_n, \bigcup_{n=0}^{\infty} G_n\right)$ is a bijective function.

Corollary (V).

Let A, B be sets.

Let $\{C_n\}_{n=0}^{\infty}$, $\{D_n\}_{n=0}^{\infty}$ be infinite sequences of subsets of A, B respectively.

Suppose that for any $n \in \mathbb{N}$, $C_n \sim D_n$.

Also suppose that for any $j, k \in \mathbb{N}$, if $j \neq k$ then $C_j \cap C_k = \emptyset$ and $D_j \cap D_k = \emptyset$.

Then $\bigcup_{n=0}^{\infty} C_n \sim \bigcup_{n=0}^{\infty} D_n$.

6. **Example** (γ).

$\mathbb{N} \sim \mathbb{N}^2$.

Remark. Hence, by Theorem (I) and the result in Example (β), we have $\mathbb{N}^m \sim \mathbb{N}$ and $\mathbb{Z}^m \sim \mathbb{Z}$ for any $m \in \mathbb{N}^*$.

(a) *Idea.*

Break up each of $\mathbb{N}$, $\mathbb{N}^2$ into many many parts.

Match the parts with bijective functions.

Then 'glue up' these bijective functions to obtain a bijective function from $\mathbb{N}$ to $\mathbb{N}^2$.

There are many ways to do it.

(b) *Correspondence 1.* $\cdots$

(c) *Correspondence 2.* $\cdots$

(d) *Correspondence 3.* $\cdots$

Example (γ).

$\mathbb{N} \sim \mathbb{N}^2$.

(b) *Correspondence 1.*

$$
\begin{array}{c|ccc|ccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
(0,0) & (1,0) & (1,1) & (0,1) & (2,0) & (2,1) & (2,2) & (1,2) & (0,2) & \cdots
\end{array}
$$

We have constructed the bijective function $f_1 : \mathbb{N} \longrightarrow \mathbb{N}^2$ below which 'matches' the respective entries at the corresponding positions of the following 'infinite square-arrays' to each other:

$$
\left|\begin{array}{ccccccc}
0 & 1 & 4 & 9 & 16 & 25 & \cdots \\
3 & 2 & 5 & 10 & 17 & 26 & \cdots \\
8 & 7 & 6 & 11 & 18 & 27 & \cdots \\
15 & 14 & 13 & 12 & 19 & 28 & \cdots \\
24 & 23 & 22 & 21 & 20 & 29 & \cdots \\
35 & 34 & 33 & 32 & 31 & 30 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right|
\longrightarrow
\left|\begin{array}{ccccccc}
(0,0) & (1,0) & (2,0) & (3,0) & (4,0) & (5,0) & \cdots \\
(0,1) & (1,1) & (2,1) & (3,1) & (4,1) & (5,1) & \cdots \\
(0,2) & (1,2) & (2,2) & (3,2) & (4,2) & (5,2) & \cdots \\
(0,3) & (1,3) & (2,3) & (3,3) & (4,3) & (5,3) & \cdots \\
(0,4) & (1,4) & (2,4) & (3,4) & (4,4) & (5,4) & \cdots \\
(0,5) & (1,5) & (2,5) & (3,5) & (4,5) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right|
$$

Example (γ).

$\mathbb{N} \sim \mathbb{N}^2$.

(b) *Correspondence 1.*

$$\begin{array}{c|ccc|ccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
(0,0) & (1,0) & (1,1) & (0,1) & (2,0) & (2,1) & (2,2) & (1,2) & (0,2) & \cdots
\end{array}$$

We have constructed the bijective function $f_1 : \mathbb{N} \longrightarrow \mathbb{N}^2$ below which 'matches' the respective entries at the corresponding positions of the following 'infinite square-arrays' to each other:

$$\left|\begin{array}{ccccccc}
0 & 1 & 4 & 9 & 16 & 25 & \cdots \\
3 & 2 & 5 & 10 & 17 & 26 & \cdots \\
8 & 7 & 6 & 11 & 18 & 27 & \cdots \\
15 & 14 & 13 & 12 & 19 & 28 & \cdots \\
24 & 23 & 22 & 21 & 20 & 29 & \cdots \\
35 & 34 & 33 & 32 & 31 & 30 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right|
\longrightarrow
\left|\begin{array}{ccccccc}
(0,0) & (1,0) & (2,0) & (3,0) & (4,0) & (5,0) & \cdots \\
(0,1) & (1,1) & (2,1) & (3,1) & (4,1) & (5,1) & \cdots \\
(0,2) & (1,2) & (2,2) & (3,2) & (4,2) & (5,2) & \cdots \\
(0,3) & (1,3) & (2,3) & (3,3) & (4,3) & (5,3) & \cdots \\
(0,4) & (1,4) & (2,4) & (3,4) & (4,4) & (5,4) & \cdots \\
(0,5) & (1,5) & (2,5) & (3,5) & (4,5) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right|$$

Example (γ).

$\mathbb{N} \sim \mathbb{N}^2$.

(c) *Correspondence 2.*

$$
\begin{array}{c|cc|ccc|cccc|c}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
(0,0) & (1,0) & (0,1) & (2,0) & (1,1) & (0,2) & (3,0) & (2,1) & (1,2) & (0,3) & \cdots
\end{array}
$$

We have constructed the bijective function $f_2 : \mathbb{N} \longrightarrow \mathbb{N}^2$ below which 'matches' the respective entries at the corresponding positions of the following 'infinite square-arrays' to each other:

$$
\left|\begin{array}{cccccc}
0 & 1 & 3 & 6 & 10 & 15 \\
2 & 4 & 7 & 11 & 16 & \ddots \\
5 & 8 & 12 & 17 & \ddots & \\
9 & 13 & 18 & \ddots & & \\
14 & 19 & \ddots & & & \\
20 & \ddots & & & & \\
\end{array}\right|
\longrightarrow
\left|\begin{array}{ccccccc}
(0,0) & (1,0) & (2,0) & (3,0) & (4,0) & (5,0) & \cdots \\
(0,1) & (1,1) & (2,1) & (3,1) & (4,1) & (5,1) & \cdots \\
(0,2) & (1,2) & (2,2) & (3,2) & (4,2) & (5,2) & \cdots \\
(0,3) & (1,3) & (2,3) & (3,3) & (4,3) & (5,3) & \cdots \\
(0,4) & (1,4) & (2,4) & (3,4) & (4,4) & (5,4) & \cdots \\
(0,5) & (1,5) & (2,5) & (3,5) & (4,5) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right|
$$

Example (γ).

$\mathbb{N} \sim \mathbb{N}^2$.

(c) *Correspondence 2.*

$$
\begin{array}{c|cc|ccc|cccc|c}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
(0,0) & (1,0) & (0,1) & (2,0) & (1,1) & (0,2) & (3,0) & (2,1) & (1,2) & (0,3) & \cdots
\end{array}
$$

We have constructed the bijective function $f_2 : \mathbb{N} \longrightarrow \mathbb{N}^2$ below which 'matches' the respective entries at the corresponding positions of the following 'infinite square-arrays' to each other:

$$
\left|\begin{array}{cccccc}
0 & 1 & 3 & 6 & 10 & 15 \\
2 & 4 & 7 & 11 & 16 & \ddots \\
5 & 8 & 12 & 17 & \ddots & \\
9 & 13 & 18 & \ddots & & \\
14 & 19 & \ddots & & & \\
20 & \ddots & & & &
\end{array}\right|
\longrightarrow
\left|\begin{array}{ccccccc}
(0,0) & (1,0) & (2,0) & (3,0) & (4,0) & (5,0) & \cdots \\
(0,1) & (1,1) & (2,1) & (3,1) & (4,1) & (5,1) & \cdots \\
(0,2) & (1,2) & (2,2) & (3,2) & (4,2) & (5,2) & \cdots \\
(0,3) & (1,3) & (2,3) & (3,3) & (4,3) & (5,3) & \cdots \\
(0,4) & (1,4) & (2,4) & (3,4) & (4,4) & (5,4) & \cdots \\
(0,5) & (1,5) & (2,5) & (3,5) & (4,5) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right|
$$

Example (γ).

$\mathbb{N} \sim \mathbb{N}^2$.

(δ) *Correspondence 3.*

Define $g : \mathbb{N}^2 \longrightarrow \mathbb{N}\backslash\{0\}$ by $g(x, y) = 2^y(2x + 1)$ for any $x, y \in \mathbb{N}$. g is a bijective function.

g sets up the following 'exact correspondence' from $\mathbb{N}^2$ to $\mathbb{N}\backslash\{0\}$:

Ascending order

$$\left|\begin{array}{ccccccc}
(0,0) & (1,0) & (2,0) & (3,0) & (4,0) & (5,0) & \cdots \\
(0,1) & (1,1) & (2,1) & (3,1) & (4,1) & (5,1) & \cdots \\
(0,2) & (1,2) & (2,2) & (3,2) & (4,2) & (5,2) & \cdots \\
(0,3) & (1,3) & (2,3) & (3,3) & (4,3) & (5,3) & \cdots \\
(0,4) & (1,4) & (2,4) & (3,4) & (4,4) & (5,4) & \cdots \\
(0,5) & (1,5) & (2,5) & (3,5) & (4,5) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right| \longrightarrow \left|\begin{array}{lllllll}
1 & 3 & 5 & 7 & 9 & 11 & \ldots \\ \hline
2 \\ \hline
4 \\ \hline
8 \\ \hline
16 \\ \hline
32 \\ \hline
\vdots
\end{array}\right|$$

← {Divide by 2; remainder is 1.

← {Divide by 4; remainder is 2.

← {Divide by 8; remainder is 4.

← {Divide by 16; remainder is 8.

⋮

Define $h : \mathbb{N}\backslash\{0\} \longrightarrow \mathbb{N}$ by $h(w) = w - 1$ for any $w \in \mathbb{N}\backslash\{0\}$.

h is a bijective function.

Now $h \circ g$ is a bijective function from $\mathbb{N}^2$ to $\mathbb{N}$, given by

$$(h \circ g)(x, y) = 2^y(2x + 1) - 1 \quad \text{for any } x, y \in \mathbb{N}.$$

Example (γ).

$\mathbb{N} \sim \mathbb{N}^2$.

(d) *Correspondence 3.*

Define $g : \mathbb{N}^2 \longrightarrow \mathbb{N} \backslash \{0\}$ by $g(x, y) = 2^y(2x + 1)$ for any $x, y \in \mathbb{N}$. g is a bijective function.

g sets up the following 'exact correspondence' from $\mathbb{N}^2$ to $\mathbb{N} \backslash \{0\}$:

Ascending order

$$
\left|\begin{array}{ccccccc}
(0,0) & (1,0) & (2,0) & (3,0) & (4,0) & (5,0) & \cdots \\
(0,1) & (1,1) & (2,1) & (3,1) & (4,1) & (5,1) & \cdots \\
(0,2) & (1,2) & (2,2) & (3,2) & (4,2) & (5,2) & \cdots \\
(0,3) & (1,3) & (2,3) & (3,3) & (4,3) & (5,3) & \cdots \\
(0,4) & (1,4) & (2,4) & (3,4) & (4,4) & (5,4) & \cdots \\
(0,5) & (1,5) & (2,5) & (3,5) & (4,5) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right|
\xrightarrow{\text{Successive multiplication by 2}}
\left|\begin{array}{ccccccc}
1 & 3 & 5 & 7 & 9 & 11 & \cdots \\
2 & 6 & 10 & 14 & 18 & 22 & \cdots \\
4 & 12 & 20 & 28 & 36 & 44 & \cdots \\
8 & 24 & 40 & 56 & 72 & 88 & \cdots \\
16 & 48 & 80 & 112 & 144 & 176 & \cdots \\
32 & 96 & 160 & 224 & 288 & 352 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}\right|
\begin{array}{l}
\leftarrow \begin{cases}\text{Divide by 2;} \\ \text{remainder is 1.}\end{cases} \\
\leftarrow \begin{cases}\text{Divide by 4;} \\ \text{remainder is 2.}\end{cases} \\
\leftarrow \begin{cases}\text{Divide by 8;} \\ \text{remainder is 4.}\end{cases} \\
\leftarrow \begin{cases}\text{Divide by 16;} \\ \text{remainder is 8.}\end{cases} \\
\vdots
\end{array}
$$

Define $h : \mathbb{N} \backslash \{0\} \longrightarrow \mathbb{N}$ by $h(w) = w - 1$ for any $w \in \mathbb{N} \backslash \{0\}$.

h is a bijective function.

Now $h \circ g$ is a bijective function from $\mathbb{N}^2$ to $\mathbb{N}$, given by

$$(h \circ g)(x, y) = 2^y(2x + 1) - 1 \quad \text{for any } x, y \in \mathbb{N}.$$

7. **Example** (δ).

Suppose I is an interval with more than one point. Then $I \sim \mathbb{R}$.

- *Outline of argument:*

(a) Suppose I is 'finite at both ends'. Deduce:

(a1) $I \sim [0, 1]$ if I is closed.

(a2) $I \sim [0, 1)$ if I is half-closed-half-open.

(a3) $I \sim (0, 1)$ if I is open.

(b) Suppose $I \neq \mathbb{R}$ and I is not 'finite at both ends'. Deduce:

(b1) $I \sim [0, +\infty)$ if I is closed.

(b2) $I \sim (0, +\infty)$ if I is open.

(c) Deduce that $[0, 1] \sim [0, 1)$. Similarly deduce that $[0, 1) \sim (0, 1)$.

(d) Deduce that $(0, 1) \sim (0, +\infty)$. Similarly deduce that $[0, 1) \sim [0, +\infty)$.

(e) Deduce that $(0, 1) \sim \mathbb{R}$.

Example (δ).

Suppose I is an interval with more than one point. Then $I \sim \mathbb{R}$.

- *Outline of argument*:

(a) Suppose I is 'finite at both ends'. Deduce:

(a1) $I \sim [0,1]$ if I is closed.

(a2) $I \sim [0,1)$ if I is half-closed-half-open.

(a3) $I \sim (0,1)$ if I is open.

(b) Suppose $I \neq \mathbb{R}$ and I is not 'finite at both ends'. Deduce:

(b1) $I \sim [0,+\infty)$ if I is closed.

(b2) $I \sim (0,+\infty)$ if I is open.

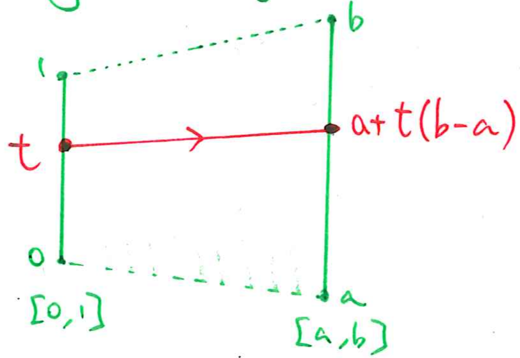
(c) $\cdots$

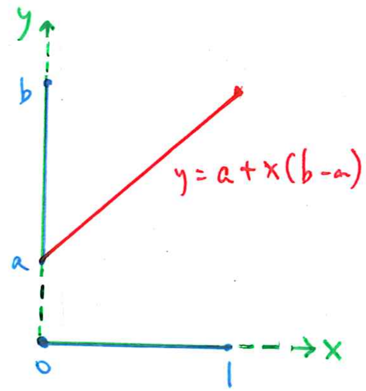
(d) $\cdots$

(e) $\cdots$

- Respective arguments for (a), (b):
Make use of 'linear functions'.

- Argument for (a1):





- How about (b1)?

Example (δ).

Suppose I is an interval with more than one point. Then $I \sim \mathbb{R}$.

- *Outline of argument:*

(a) $\cdots$

(b) $\cdots$

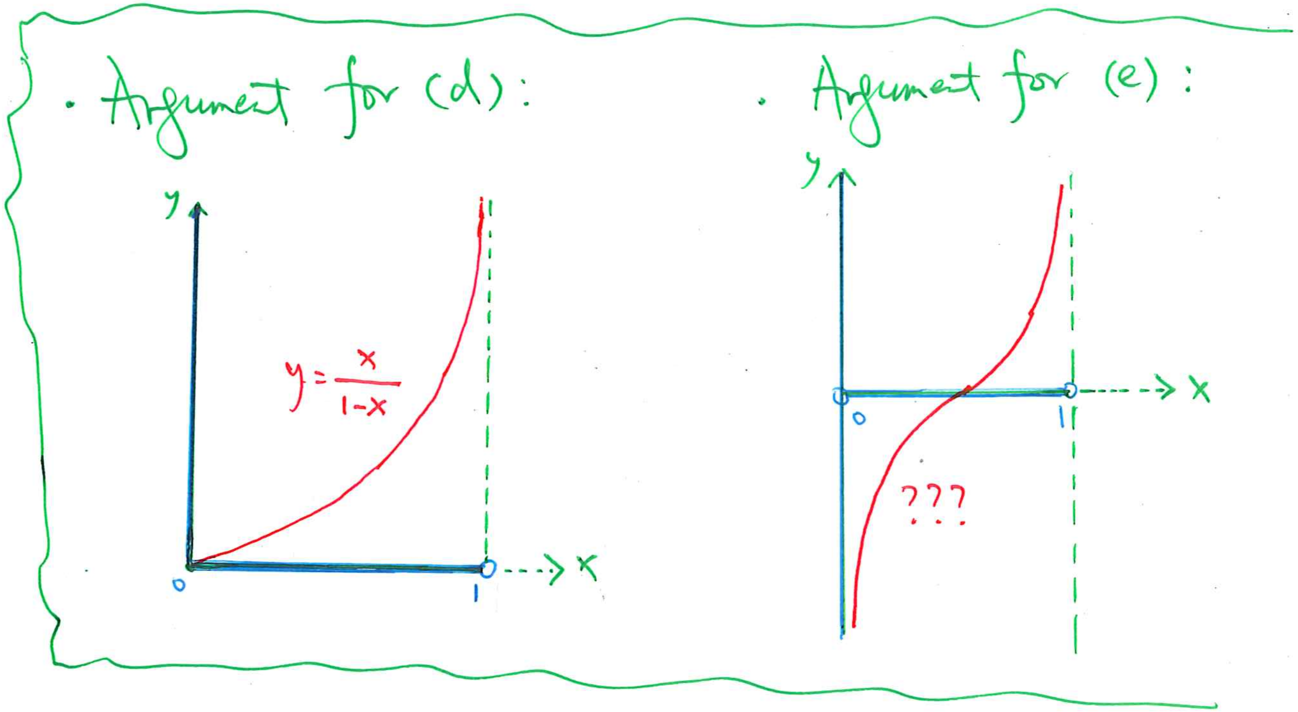
(c) Deduce that $[0,1] \sim [0,1)$. Similarly deduce that $[0,1) \sim (0,1)$.

(d) Deduce that $(0,1) \sim (0,+\infty)$. Similarly deduce that $[0,1) \sim [0,+\infty)$.

(e) Deduce that $(0,1) \sim \mathbb{R}$.

Respective arguments for (d), (e):

Make use of 'rational functions'.



Argument for (c)? Non-trivial.

Example (δ).

Argument for (c): $[0,1]\sim[0,1)$? $[0,1)\sim(0,1)$?

- *Idea.*

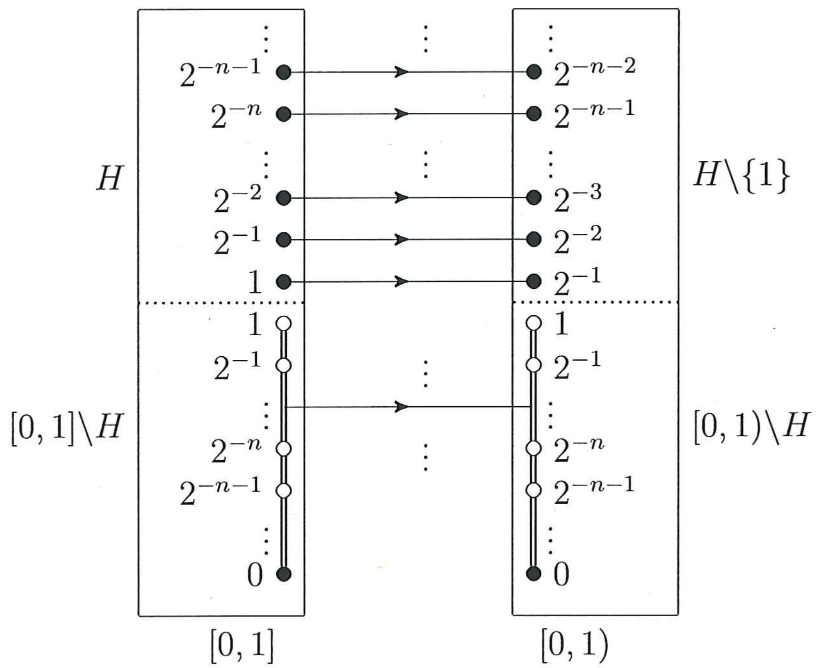
  $[0,1)$ is almost the whole of $[0,1]$ except that it 'misses' the point 1.

  Try to 'modify' the identity function from $[0,1]$ to $[0,1]$ to get a bijective function from $[0,1]$ to $[0,1)$.

- *Trick.*

  Dig many many holes in $[0,1]$, $[0,1)$ at identical positions so that after this digging, what remain of these two sets are the same set.

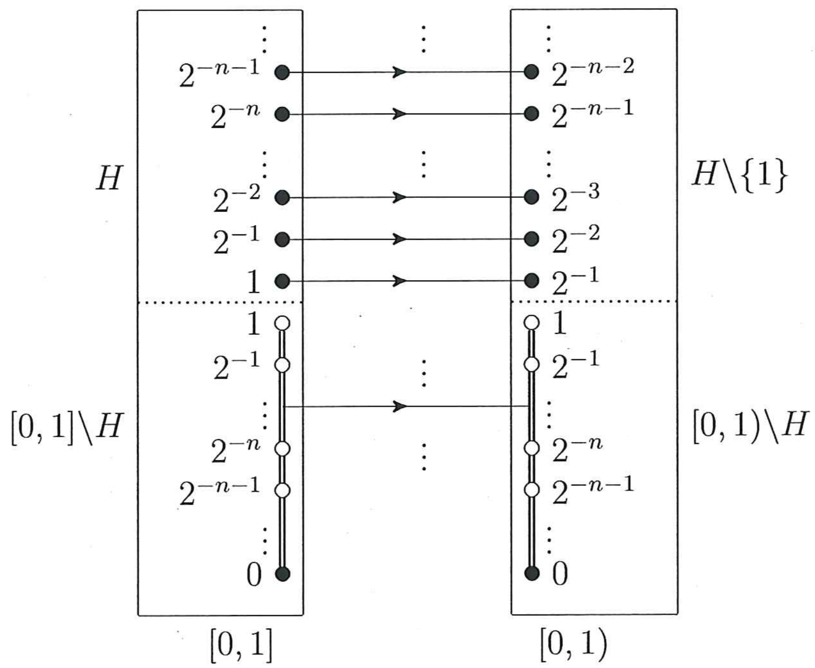
  (But what to do with the 'debris'? Don't throw them away.)



Example (δ).

Argument for (c): $[0,1]\sim[0,1)$? $[0,1)\sim(0,1)$?

- *Trick.*



- *Formal argument.*

  Define $H = \left\{ \frac{1}{2^n} \;\middle|\; n \in \mathbb{N} \right\}$.

  Note that $[0,1]\backslash H = [0,1)\backslash H$.

  Define $F_1 = \{(x,x) \mid x \in [0,1]\backslash H\}$,
  $F_2 = \left\{(x, \frac{x}{2}) \mid x \in H\right\}$ and
  $F = F_1 \cup F_2$.

  Verify that $f_1 = ([0,1]\backslash H, [0,1)\backslash H, F_1)$, $f_2 = (H, H\backslash\{1\}, F_2)$ are bijective functions.

  Define $f = ([0,1],[0,1),F)$. f is a relation. f is a bijective function according to the 'Glueing Lemma'.

8. **Example** (ϵ).

Suppose A is a set. Then $\mathfrak{P}(A)\sim\mathsf{Map}(A,\{0,1\})$.

(a) *Idea* (through one example).

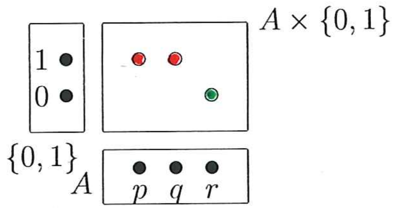
Let $A=\{p,q,r\}$, where p,q,r are pairwise distinct.

'Light bulb' analogy:

* Imagine:
  p,q,r are points on the plane.
  One light-bulb is fixed at each point.

* Name any subset S of A.
  Turn on the lights at the elements of S.
  Leave others alone.
  This give an 'overall state' of the 'light bulbs' in A according to what S is.

* For instance, when $S=\{p,q\}$, we have:



$\leftarrow$ p, q: 'on'.
r: 'off'.

Here '0' stands for off, and '1' stands for 'on'.

* Such a diagram is in fact a graph of the function from A to $\{0,1\}$.
  (When $S=\{0,1\}$, the function concerned assigns p,q,r to $1,1,0$ respectively.)

Example (ϵ).

Suppose A is a set. Then $\mathfrak{P}(A) \sim \mathsf{Map}(A, \{0, 1\})$.

(a) *Idea* (through one example).

Let $A = \{p, q, r\}$, where p, q, r are pairwise distinct.

'Light bulb' analogy:

$*$ *Observation.*

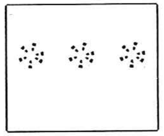
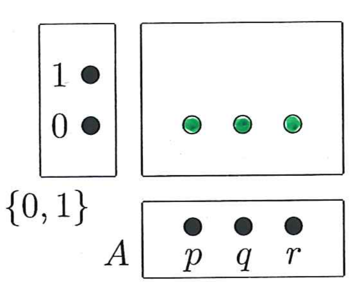
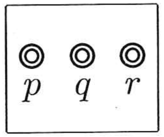
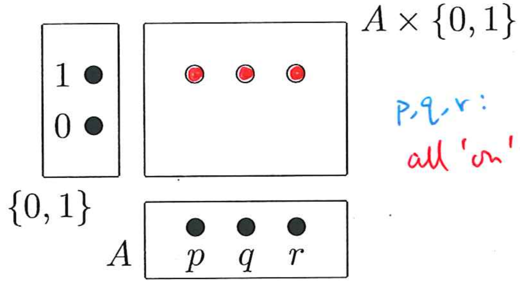
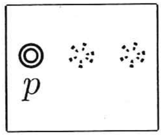
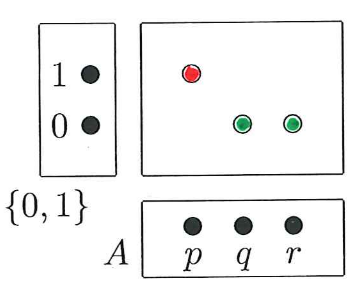
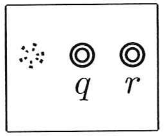
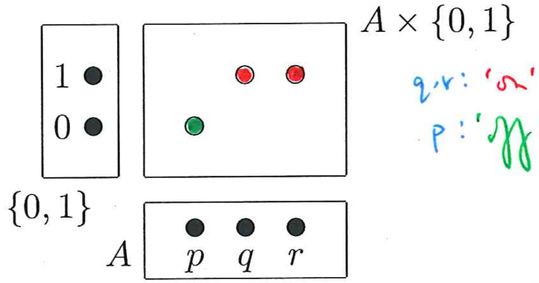
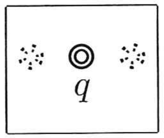
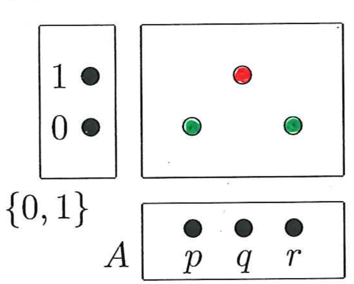
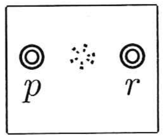
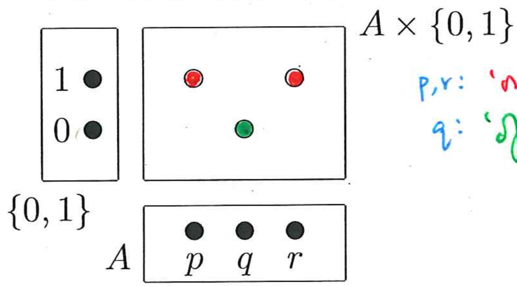
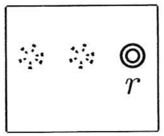
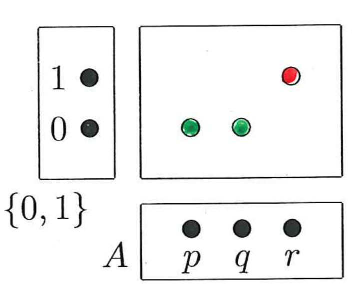
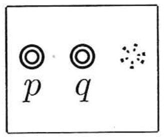
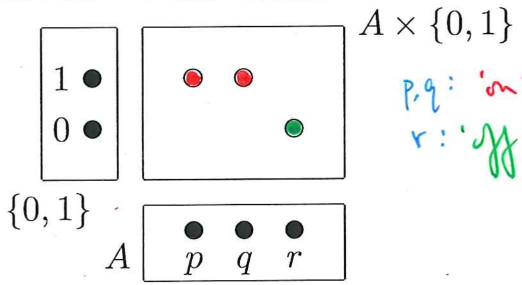
Each individual element of $\mathfrak{P}(A)$ corresponds to exactly one 'overall state' of the "light-bulbs" in A.

So we have a 'natural' 'exact correspondence' between

the subsets of A

and

the functions from A to $\{0, 1\}$ (as visualized by their respective graphs).

Subsets of A	Functions from A to $\{0,1\}$, represented by their graphs	Subsets of A	Functions from A to $\{0,1\}$, represented by their graphs
 $\emptyset$	 p,q,r: all 'off'.	 $\{p,q,r\}$	 p,q,r: all 'on'.
 $\{p\}$	 p: 'on'. q,r: 'off'.	 $\{q,r\}$	 q,r: 'on'. p: 'off'.
 $\{q\}$	 q: 'on'. p,r: 'off'.	 $\{p,r\}$	 p,r: 'on'. q: 'off'.
 $\{r\}$	 r: 'on'. p,q: 'off'.	 $\{p,q\}$	 p,q: 'on', r: 'off'.

Example (ϵ).

Suppose A is a set. Then $\mathfrak{P}(A) \sim \mathsf{Map}(A, \{0,1\})$.

(b) *Formal argument.*

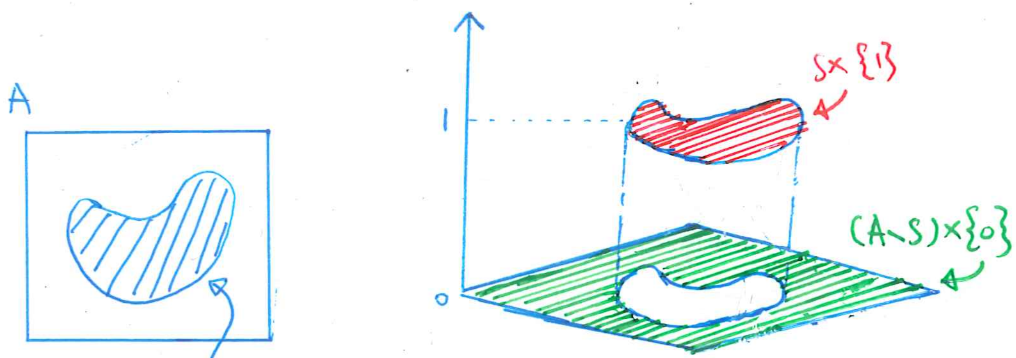
Suppose A is a set. Then $A = \emptyset$ or $A \neq \emptyset$.

If $A = \emptyset$ then $\mathfrak{P}(A) = \{\emptyset\}$ and $\mathsf{Map}(A, \{0,1\}) = \{(\emptyset, \{0,1\}, \emptyset)\}$. [Done.]

From now on suppose $A \neq \emptyset$.

For each $S \in \mathfrak{P}(A)$, define the function $\chi_S^A : A \longrightarrow \{0,1\}$ by

$$\chi_S^A(x) = \begin{cases} 1 & \text{if} \quad x \in S \\ 0 & \text{if} \quad x \in A \backslash S. \end{cases}$$



$S \overset{f}{\longmapsto} \chi_S^A$ with the graph above

Define the function

$$f : \mathfrak{P}(A) \longrightarrow \mathsf{Map}(A, \{0,1\})$$

by

$$f(S) = \chi_S^A \quad \text{for any } S \in \mathfrak{P}(A).$$

Verify that f is bijective. (Fill in the detail.)

Remark. χ_S^A is called the **characteristic function of the set S in the set A.**

9. **Example** (ζ).

$\mathsf{Map}(\mathbb{N}, \{0,1\}) \sim (\mathsf{Map}(\mathbb{N}, \{0,1\}))^2$.

(a) *Idea.*

Each element of $\mathsf{Map}(\mathbb{N}, \{0,1\})$ is a function from $\mathbb{N}$ to $\{0,1\}$, and hence is an infinite sequence in $\{0,1\}$.

Is there any natural 'exact correspondence' between infinite sequences in $\{0,1\}$ and ordered pairs of such sequences?

What can we say about the function from $\mathsf{Map}(\mathbb{N}, \{0,1\})$ to $(\mathsf{Map}(\mathbb{N}, \{0,1\}))^2$ defined by

$$(a_0, a_1, a_2, a_3, a_4, a_5, \cdots) \longmapsto ((a_0, a_2, a_4, \cdots), (a_1, a_3, a_5, \cdots))$$

for each infinite sequence $\{a_n\}_{n=0}^{\infty}$ in $\{0,1\}$?

- *Formal argument.* Exercise.

Remarks. More generally, we have:

(a) $\mathsf{Map}(\mathbb{N}, \{0,1\}) \sim (\mathsf{Map}(\mathbb{N}, \{0,1\}))^n$ for any $n \in \mathbb{N} \backslash \{0\}$.

(b) $\mathsf{Map}(\mathbb{N}, B) \sim (\mathsf{Map}(\mathbb{N}, B))^n$ for any $n \in \mathbb{N} \backslash \{0\}$, whenever B is a non-empty set.