MATH1050 Anthology on definitions for the notion of 'function'

A. Euler's *Introductio Analysin Infinitorum* [1748] (translated by Blanton [1988])

CHAPTER I
ON FUNCTIONS IN GENERAL

1. A constant quantity is a determined quantity which always keeps the same value. ...
2. A variable quantity is one which is not determined or is universal, which can take on any value. ...
3. A variable quantity is determined when some definite value is assigned to it. ...
4. A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities. Hence every analytic expression, in which all component quantities except the variable z are constants, will be a function of that z; thus $a + 3z$; $zz - 4z^2$; $az + b\sqrt{a^2 - z^2}$; c^z; et cetera are functions of z. ...

B. Euler's *Institutiones Calculi Differentialis* [1755] (translated by Blanton [2000])

PREFACE

In order that this difference between constant quantities and variables might be clearly illustrated, let us consider a shot fired from a cannon with a charge of gunpowder. ... There are many quantities involved here: First, there is the quantity of gunpowder; then, the angle of elevation of the cannon above the horizon; third, the distance travelled by the shot; and fourth, the length of time the shot is in the air. ... At the same time, it must be understood from this that in this business the thing that requires the most attention is how the variable quantities depend on each other. When one variable changes, the others necessarily are changed. For example, in the former case considered, the quantity of powder remains the same, and the elevation is changed; then the distance and duration of the flight are changed. Hence, the distance and duration are variables that depend on the elevation; if this changes, then the other also change at the same time. In the latter case, the distance and duration depend on the quantity of charge of powder, so that a change in the charge must result in certain changes in other variables.

Those quantities that depend on others in this way, namely, those that undergo a change when others change, are called functions of these quantities. This definition applies rather widely and includes all ways in which one quantity can be determined by others. Hence, if x designates the variable quantity, all other quantities that in any way depend on x or are determined by it are called its functions. Examples are x^2, the square of x, or any other powers of x, and indeed, even quanitites that are composed with these powers in any way, even transcendentals, in general, whatever depends on x in such a way that when x increases or decreases, the function changes. ...

C. Lagrange's *Théorie des fonctions analytiques* [Second Edition 1813] (Stedall's *Mathematics Emerging* [2008])

INTRODUCTION
Functions in general. ...

One calls a function of one or more quantities every expression of calculus in which the quantities enter in any manner, mixed or not with other quantities that one regards as having given and fixed values, as long as the quantities in the function may receive every possible value. Thus, in functions, one considers only the quantities that one supposes variable, without any regard to constants which may be mixed with them.

The word function was employed by the first analysts to denote, in general, powers of the same quantity. Since then one has understood this word to mean every quantity formed in any manner whatever from another quantity. Leibniz and the Bernoullis employed it in this general sense, and it is today generally accepted. ...

D. Cauchy's *Cours d'Analyse de l'École Royale Polytechnique* [1821] (translatedd by Bradley and Sandifer [2009])

CHATPER I
ON REAL FUNCTIONS

1.1 General Considerations on Functions.

When variable quantities are related to each other such that the value of one of the variables being given one can find the values of all the other variables, we normally consider these various quantities to be expressed by means of the one among them, which therefore takes the name the independent variable. The other quantities expressed by means of the independent variables are called functions of that variable.

When variable quantities are related to each other such that the value of some of them being given one can find all of the others, we consider these various quantities to be expressed by means of several among them, which therefore take the name independent variables. The other quantities expressed by means of the independent variables are called functions of those same variables.

The various expressions that are used in algebra and trigonometry, when they involve variables that are considered to be independent, are also functions of these same variables. And so, for example, $\log(x), \sin(x)$, ... are functions of the variable x, while $x + y$, x^y, xyz, ... are functions of the variables x and y, or of x, y and z, ...

When the functions of one or several variables are directly expressed, as in the preceding examples, by means of those same variables, they are called explicit functions. But when they are given only as relations among the functions and the variables, that is to say the equations that the quantities must satisfy, as long as the equations are not solved algebraically, the functions are not expressed directly by means of the variables, then they are called implicit functions. To make them explicit, it suffices to solve, when it is possible, the equations that determine them. For example, when y is given by the equation $\log(y) = x$, then it is an implicit function of x. If we let A be the base of the system of logarithms being considered, the same function made explicit by solving the given equation will be $y = A^x$.

When we want to denote an explicit function of a single variable x or of several variables, $x, y, z, ...$, without specifying the nature of that function, we use one of the notations, $f(x)$, $F(x)$, $\phi(x)$, $\chi(x)$, $\psi(x)$, $\varpi(x)$, ..., $f(x, y, z, ...)$, $F(x, y, z, ...)$, $\phi(x, y, z, ...)$, ...

E. Fourier's *Théorie analytique de la chaleur* [1822] (Stedall's *Mathematics Emerging* [2008])

> ... In general, the function $f(x)$ represents a series of values, or ordinates, each of which is arbitrary. The abscissa may take an infinite number of values, and there are the same number of ordinates $f(x)$. All have for the time being numerical values, positive, negative, or zero. one does not at all suppose that the ordinates are subject to a common law: they succeed each other in any manner whatsoever, and each is given as if it were a single quantity. ...

F. Weierstrass's lectures on the differential calculus [1861] (Calinger's *Classics of Mathematics* [1982])

> ... Two variable magnitudes may be related in such a way that to every definite value of one there corresponds a definite value of the other; then the latter is called a function of the former. This relationship may extend to several variable magnitudes; accordingly one distinguishes functions with one and with several variable magnitudes. If to one value of the one variable magnitude there always corresponds only one value of another, then the latter is called an unambiguous function and single-valued function of the former. If to one value of the one magnitude there correspond several values of another, then the latter is called a multi-valued function of the former. The criterion of a function is that the one variable magnitude changes in general by a definite amount as soon as a definite change of the other one is assumed. ...

G. Dedekind's *Was sind und was sollen die Zahlen?* [1888] (translated by Beman [1901])

> ... By a transformation[1] ϕ of a system[2] S we understand a law according to which to every determinate element s of S there belongs a determinate thing which is called the transform of s and denoted by $\phi(s)$; we say also that $\phi(s)$ corresponds to the element s, that $\phi(s)$ results or is produced from s by the transformation ϕ, that s is transformed into $\phi(s)$ by the transformation ϕ. ...

[1]Read function.

[2]Read set.

H. Goursat's *Cours d'analyse mathématique* [1902] (translated by Hedrick [1904])

CHAPTER I
DERIVATIVES AND DIFFERENTIALS
I. FUNCTIONS OF A SINGLE VARIABLE

1. Limits. ...
2. Functions. When two variable quantities are so related that the value of one of them depends upon the value of the other, they are said to be functions of each other. If one of them be supposed to vary arbitrarily, it is called the independent variable. Let this variable be denoted by x, and let us suppose, for example, that it can assume all values between two given numbers a and b ($a < b$). Let y be another variable, such that to each value of x between a and b, and also for the values a and b themselves, there corresponds one definitely determined value of y. Then y is called a function of x, defined in the interval (a, b); and this independence is indicated by writing the equation $y = f(x)$. For instance, it may happen that y is the result of certain arithmetical operations performed upon x. ... A function may also be defined graphically. ... In short, any absolutely arbitrary law may be assumed for finding the value of y from that of x. The word function, in its most general sense, means nothing more nor less than this: to every value of x corresponds a value of y.

I. Hardy's *A course of Pure mathematics* (Tenth Edition) [1952]

CHAPTER II
FUNCTIONS OF REAL VARIABLES

20. **The idea of a function.** Suppsoe that x and y are two continuous real variables, which we may suppose to be represented by distances $A_0P = x$, $B_0Q = y$ measured from fixed points A_0, B_0 along two straight lines Λ, M. And let us suppose that the positions of the points P and Q are not independent, but are connected by a relation which we can imagine expressed as a relation between x and y; so that, when P and x are known, Q and y are also known. We might, for example, suppose that $y = x$, or $2x$, or $\frac{1}{2}x$, or $x^2 + 1$. In all of these cases the value of x determines that of y. Or again we might suppose that the relation between x and y are given, not by means of an explict formula for y in terms of x, but by means of a geometrical construction which enables us to determine Q when P is known.

    In these circumstances y is said to be a *function* of x. This notion of functional dependence of one variable upon another is perhaps the most important in the whole range of higher mathematics. In order to enable the reader to be certain that he understands it clearly, we shall, in this chapter, illustrate it by means of a large number of examples.

    But before we proceed to do this, we must point out that the simple examples of functions mentioned above possess three characteristics which are by no means involved in the general idea of a function, viz.:

    (1) y is detemined *for every value of* x;
    (2) to each value of x for which y is given corresponds *one and only one value of* x;
    (3) the relation between x and y is expressed by means of *an analytical formula*, for which the value of y corresponding to a given value of x can be calculated by direct substitution of the latter.

    It is indeed the case that these particular characteristics are possessed by many of the most important functions. But the consideration of the following examples will make it clear that they are by no means essesntial to a function. All that is essential is that there should be some relation between x and y such that to some values of x at any rate correspond values of y.

    Examples X.

    1. Let $y = x$ or $2x$ or $\frac{1}{2}x$ or $x^2 + 1$. Nothing further need be said at present about cases such as these.
    2. Let $y = 0$ whatever be the value of x. Then y is a function of x, for we can give x any value, and the corresponding value of y (viz. 0) is known. In this case the functional relation makes the same value of y correspond to all values of x. The same would be true were y equal to 1 or $-\frac{1}{2}$ or $\sqrt{2}$ instead of 0. Such a function of x is called a *constant*.

3. Let $y^2 = x$. Then if x is positive this equation defines *two* values of y corresponding to each value of x, viz. $\pm\sqrt{x}$. If $x = 0$, $y = 0$. Hence to the particular value 0 of x corresponds *one* and only one value of y. But if x is negative there is *no* value of y which satisfies the equation. That is to say, the function y is not defined for negative values of x. This function therefore possesses the characteristic (3), but neither (1) nor (2).
4. Consider a volume of gas maintained at a constant temperature and constained in a cylinder closed by a sliding piston.
Let A be the area of the cross-section of the piston and W is weight. The gas, held in a state of compression by the piston, exerts a certain pressure p_0 per unit of area on the piston, which balances the weight W, so that
$$W = Ap_0.$$
Let v_0 be the volume of the gas when the system is thus in equilibrium. If additional weight is placed upon the piston the latter is forced downwards. The volume (v) of the gas diminishes; the pressure (p) which it exerts upon the unit area of the piston increases. Boyle's experimental law asserts that the product of p and$v$ is very nearly constant, a correspondance which, if exact, would be represented by an equation of the type
$$pv = a \quad \cdots\cdots\cdots\cdots (i),$$
where a is a number which can be determined approximately by experiment.
Boyle's law, however, only gives a reasonable approximation to the facts provided the gas is not compressed too much. When v is decreased and p increased beyond a certain point, the relation between them is no longer expressed with tolerable exactness by the equation i. It is known that a much better approximation to the true relation can then be found by means of what is known as 'van der Waal's law', expressed by the equation
$$\left(p + \frac{\alpha}{v^2}\right)(v - \beta) = a \quad \cdots\cdots\cdots\cdots (ii),$$
where α, β, γ are numbers which can also be determined approximately by experiment.
Of course the two equations, even taken together, do not give anything like a complete account of the relation between p and v. This relation is no doubt in reality much more complicated, and its form changes, as v varies, from a form nearly equivalent to (i) to a form nearly equivalent to (ii). But, from a mathematical point of view, there is nothing to prevent us from contemplating an ideal state of things in which, for all values of v not less than a certain value V, (i) would be exactly true, and (ii) exactly true for all value of v less than V. And then we might regard the two equations as together defining p as a function v. It is an example of a function which for some values of v is defined by one formula and for other values of v is defined by another.
This function possess the characteristic (2); to anty value of v only one value of p corresponds; but it does not possess (1). For p is not defined as a function of v for negative values of v; a 'negative volume' means nothing, and negative values of v are irelevant.
5. Suppose that a perfectly elastic ball is dropped (without rotation) from a height $\frac{1}{2}gr^2$ on to a fixed horizontal plane, and rebounds continually.
The ordinary formulae of elementary dynamics, with which the reader is probably familiar, show that $h = \frac{1}{2}gt^2$ if $0 \le t \le r$, $h = \frac{1}{2}(2r - t)^2$ if $r \le t \le 3r$, and generally
$$h = \frac{1}{2}g(2nr - t)^2$$
if $(2n-1)r \le t \le (2n+1)r$, h being the depth of the ball, at time t, below its original position. Here also h is a function of t which is only defined for positive values of t.
6. Suppose that y is defined as being *the largest prime factor of* x. This is an instance of a definition which only applies to a particular class of values of x, viz. *integral* values . 'The largest prime factor' of $\frac{11}{3}$ of of $\sqrt{2}$ or of π' means nothing, and so our defining relation fails to define for such values of x as these. Thus this function does not possess the characteristic (1). It possesses (2), but not (3), since there is no simple formula which expresses y in terms of x.

7. Let y be defined as *the denominator of x when x is expressed in its lowest terms.* This is an example of a function which is defined if and only if x is rational. Thus $y = 7$ if $x = -11/7$, but y is not defined for $x = \sqrt{2}$.

21. **The graphical representation of functions.** ...
22. **Polar coordinates.** ...
23. **Further examples of functions and their graphical representation.** The examples which follow will give the reader a better notion of the infinite variety of possible types of functions.
    A. **Polynomials.** ...
24. B. **Rational functions.** ...
25. The graphical study of rational functions depends, even more than that of polynomials, on the methods of the differential calculus. ...
26. C. **Explicit algebraic functions.** ...
27. D. **Implicit algebraic functions.** ...
28. **Transcendental functions.** ...
    E. **The direct and inverse trigonometric or circular functions.** ...
29. F. **Other classes of transcendental functions.** Next in importance to the trigonometrical functions come the exponential and logarithmic functions, which will be discussed in Chs. IX and X. But these functions are beyond our range at present. And most of the other classes of transcendental functions whose properties have been studied, such as the elliptic functions, Bessel's and Legendre's functions, gamma-functions, and so forth, lie altogether beyond the scope of this book. There are however some elementary types of functions which, though of much less importance theoretically than the rational, algebraical, or trigonometrical functions, are particulary instructive as illustrations of the possible varieties of the functional relation. ...

CHAPTER X

THE GENERAL THEORY OF THE LOGARITHMIC, EXPONENTIAL, AND CIRCULAR FUNCTIONS

227. **Functions of a complex variable.** In Ch. III we defined the complex variable

$$z = x + iy,$$

and we considered a few simple properties of some classes of expressions involving z, such as the polynomial $P(z)$. It is natural to describe such expressions as *functions* of z, and in fact we did describe the quotient $P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials, as a 'rational function'. We have however given no general definition of what is meant by a function of z.

It might seem natural to define a function of z in the same way as that in which we defined a function of the real variable x, i.e. to say that Z is a function of z if there is any relation between z and Z in virtue of which a value of values of Z corresponds to some or all values of z. But it will be found, on closer examination, that this definition is not one from which any profit can be derived. For if z is given, so are x and y, and conversely: to assign a value of z is just the same thing as to assign a pair of values of x and y. Thus a 'function of z', according to the definition suggested, is merely a *complex function*

$$f(x, y) + ig(x, y),$$

of two real variables x and y. For example,

$$x - iy, \quad xy, \quad |z| = \sqrt{x^2 + y^2}, \quad \mathrm{am}(z) = \arctan(y/x)$$

are 'functions of z'. The definition, although quite legitimate, is futile because it does not really define a new idea at all.

It is therefore more convenient to use the expression 'function of the complex variable z' in a more resstricted sense, or in other words to pick out, from the general class of complex functions of the two real variables x and y, a special class to which the expression shall be restricted. If we were to explain how this selection is made, and what are the characteristic properties of the special class of functions selected, we should be led fra beyond the limits of this book. We shall therefore not attempt to give any general definitions, but shall confine ourselves entirely to special functions defined directly.

32. One-many relations or functions

We will now deal in some detail with another particularly important category of relations. A relation R is called a ONE-MANY or FUNCTIONAL RELATION or simply a FUNCTION if, to every thing y there corresponds at most one thing x such that xRy; in other words, if the formulas:

$$xRy \quad \text{and} \quad zRy$$

always imply the formula:

$$x = z.$$

The successors with respect to the relation R, that is, thos things y for which there acutally are things x such that

$$xRy,$$

are the ARGUMENT VALUES, the predecessors are the FUNCTION VALUES or, simply, the VALUES OF THE FUNCTION R. Let R be an arbitary function, y any one of its argument values; the unique value x of the function corresponding to the value y of the argument we will denote by the symbol "R(y)"; consequently we replace the formula:

$$xRy$$

by:

$$x = R(y).$$

It has become the custom, especially in mathematics, to use, not the variables "R", "S", ..., but other letters such as "f", "g", ... to denote functional relations, so that we find formulas like these:

$$x = f(y), \quad x = g(y), \cdots ;$$

the formula:

$$x = f(y),$$

for instance, is read as follows:

the function f assigns (or *correlates*) *the value x to the argument value y*

or

x is that value of the function f which corresponds to (or is correlated with) the argument value y.

(There is also another custom, of using the variable "x" for denoting the argument value and the variable "y" for denoting the value of the function. We shall not adhere to this custom, and continue to use "x" and "y" in the opposite order, because this is more convenient in connection with the general notation used in the theory of relations.)

In many elementary textbooks of algebra a definition of the concept of a function is to be found that is quite different from the definition adopted here. The functional relation is there characterized as a relation between two "variable" quantities or numbers: the "independent variable" and the "dependent variable", which depend upon each other in so far as a change of the first effects a change of the second. Definitions of this kind should no longer be employed today, since they are incapable of standing up to anly logical criticism; they are the remains of a period which one tried to distinguish between "constant" and "variable" quantitites (cf. Section 1). He who desires to comply with the requirements of contemporary science and yet does not wish to break away completely from tradition, may, however, retain the old terminology and use, beside the terms "argument value" and "function value", the expressions "value of the independent variable" and "value of the dependent variable".

The simplest example of a functional relation is represented by the ordinary relation of identity. As an example of a funciton from everyday life let us take the relation expressed by the sentential function:

x is father of y.

It is a functional relation, since, to every person y, there exists but one person x who is father of y. In order to indicate the functional character of this relation, we insert the word "the" in the above formulation:

$$x \textit{ is the father of } y,$$

instead of which we might also write:

$$x \textit{ is identical with the father of } y.$$

Such an alteration of the original expression, involving the insertion of the definite article, serves, in ordinary language, exactly the same purpose as the transition from the formula:

$$xRy$$

to the formula

$$x = R(y)$$

in our symbolism.

The concept of a function plays a most important role in the mathematical sciences. There are whole branches of higher mathematics devoted exclusively to the study of certain kinds of functional relations. But also in elementary mathematics, especially in algebra and trigonometry, we find an abundance of functional relations. Examples are the relations expressed by such formulas as:

$$\begin{aligned} x + y &= 5, \\ x &= y^2, \\ x &= \log_{10} y, \\ x &= \sin y, \end{aligned}$$

and many others. Let us consider the second of these formulas more closely. To every number y, there corresponds only one number x such that $x = y^2$, so that the formula really does represent a functional relation. Argument values of this function are arbitrary numbers, values of the function, however, only non-negative numbers. If we denote this function by the symbol "f", the formmula:

$$x = y^2$$

assumes the form:

$$x = f(y).$$

Evidently "x" and "y" may here be replaced by symbols designating definite numbers. Since, for instance,

$$4 = (-2)^2,$$

it may be asserted that

$$4 = f(-2);$$

thus, 4 is the value of the function f corresponding to the argument value -2.

On the other hand, and again in elementary mathematics already, we encounter numerous relations which are not functions. For example, the relation of being smaller is certainly not a function, since, to every number y, there are infinitely many numbers x such that

$$x < y.$$

Nor is the relation between the numbers x and y expressed by the formula:

$$x^2 + y^2 = 25$$

a functional relation since, to one and the same number y, there may correspond two different numbers x for which the formula is valid; corresponding to the number 4, for instance, we have both the numbers 3 and -3. It may be noted that relations between numbers which, lik the one just considered, are expressed by equations and correlate with one number y two or more numbers x are sometiems called in mathematics two- or many-valued functions (in opposition to single-valued functions, that is, to functions in the ordinary meaning). It seems, however, inexpedient — at least on an elementary level — to denote such relations as functions, for this only tends to blot out the essential difference between the notion of a function and the more general one of a relation.

Functions are of particular significance as far as the application of mathematics to the empirical sciences is concerned. Whenever we inquire into the dependence between two kinds of quantities occuring in the external world, we strive to give this dependence the form of a mathematical formula, which would permit us to determine exactly the quantity of the one kind by the corresponding quantity of the other; such a formula always represents some functional relation between the quantitites of two kinds. As an example let us mention the well-known formula from physics:

$$s = 16.1t^2$$

expressing the dependence of the distance s, covered by a freely falling body, upon the time t of its fall (the distance being measured in feet and the time in seconds). ...

33. One-to-one relations or biunique functions, and one-to-one correspondences

Among the functional relations particular attention should be paid to the so-called ONE-ONE RELATIONS or BIUNIQUE FUNCTIONS, that is, to those functional relations in which not only every argument value y only one function value x is correlated, but also converselyonly one argument value y corresponds to every value x of the function; they might also be defined as those relations which have the property that their converses (cf. Section 28) as well as the relations themselves are one-many.

If f is a biunique funciton, K an arbitrary class of its argument values, and L the class of function values correlated with the elements of K, we say that the function f MAPS THE CLASS K ON THE CLASS L IN A ONE-TO-ONE MANNER, or that it ESTABLISHES A ONE-TO-ONE CORRESPONDENCE BETWEEN THE ELEMENTS OF K AND L.

Let us consider a few examples. Suppose we have a half-line issuing from the point O, with a segment marked off indicating the unit of length. Further let Y be any point on the half-line. Then the segment OY can be measured, that is to say, one can correlate with it a certain non-negative number x called the length of the segment. Since this number depends exclusively on the position of the point Y, we may denoted it by the symbol "$f(Y)$"; we consequently have:

$$x = f(Y).$$

But conversely, to every non-negative number x, we may also construct a uniquely determined segment OY on the half-line under consideration, whose length equals x; in other words, to every x, there corresponds exactly one point Y such that

$$x = f(Y).$$

The function f is, therefore, biunique; it establishes a one-to-one correspondence between the points of the half-line and the non-negative numbers (and it would be equally simple to set up a one-to-one correspondence between the points of the entire line and all real numbers). Another example is supplied by the relation expressed by the formula:

$$x = -y.$$

This is a biunique function since, to every number x, there is only one number y satisfying the given formula; it can be seen at once that this function maps, for instance, the set of all positive numbers on the set of all negative numbers in a one-to-one manner. As a last example let us consider the relation expressed by the formula:

$$x = 2y$$

under the assumption that the symbol "y" here denotes natural numbers only. Again we have a biunique function; it correlates with every natural number y and even number $2y$; and vice versa — to every even natural number x there corresponds just one number y such that $2y = x$, namely, the number $y = \frac{1}{2}x$. The function thus establishes a one-to-one correspondence between arbitrary natural numbers and even natural numbers. — Numerous examples of biunique functions and one-to-one mappings can be drawn from the field of geometry (symmetric, collinear mappings, and so on). ...

K. Suppes' *Axiomatic Set Theory* [1960]

Functions. Since the eighteenth century, clarification and generalization of the concept of a function have attracted much attention. Fourier's representation of "arbitrary" functions (actually piecewise continuous ones) by trigonometric series encountered much opposition; and later when Weierstrass and Riemann gave examples of continuous functions without derivatives, mathematicians refused to consider them seriously. Even today many textbooks of the differential and integral calculus do not give a mathematically satisfactory definition of functions. An exact and completely general definition is immediate within our set-theoretical framework. A function is simply a many-one relation, that is, a relation which to any element in its domain relates exactly one element in its range. (Of course, distinct elements in the domain may be related to the same element in the range.) The formal definition is obvious.

Definition 39.
f is a function $\longleftrightarrow$ f is a relation $\wedge$ $(\forall x)(\forall y)(\forall z)(xfy \wedge xfz \longrightarrow y = z)$.

L. Halmos' *Naive Set Theory* [1960]

SECTION 7
RELATIONS
Using ordered pairs, we can formulate the mathematical theory of relations in set-theoretic language. By a relation we mean here something like marriage (between men and women) or belonging (between elements and sets). More explicitly, what we shall call a relation is sometimes called a *binary* relation. An example of a ternary relation is parenthood for people (Adam and Eve are the parents of Cain). In this book we shall have no occasion to treat the theory of relations that are ternary, quaternary, or worse.

Look at any specific relation, such as marriage for instance, we might be tempted to consider certain ordered pairs (x, y), namely just those for which x is a man, y is a woman, and x is married to y. We have not yet seen the definition of the general concept of a relation, but it seems plausible that, just as in this marriage example, every relation should uniquely determine the set of all those ordered pairs for which the first coordinate does stand in that relation to the second. If we know the relation, we know the set, and better yet, if we know the set, we know the relation. If, for instance, we were presented with the set of ordered pairs of people that corresponds to marriage, then, even if we forgot the definition of marriage, we could always tell when a man x is married to a woman y and when not; we would just have to see whether the ordered pair (x, y) does or does not belong to the set.

We may not know what a relation is, but we do know what a set is, and the preceding considerations establish a close connection between relations and sets. The precise set-theoretic treatment of relations takes advantage of that heuristic connection; the simplest thing to do is to define a relation to be the corresponding set. This is what we do; we hereby define a *relation* as a set of ordered pairs. Explicitly; a set R is a relation if each element of R is an ordered pair; this means, of course, that if $z \in R$, then there exist x and y so that $z = (x, y)$. If R is a relation, it is sometimes convenient to express the fact that $(x, y) \in R$ by writing

$$xRy$$

and saying, as in everyday language, that x stands in the relation R to y. ...

In the preceding section we saw that associated with every set R of ordered pairs there are two sets called the projections of R onto the first and second coordinates. In the theory of relations these sets are known as the *domain* and the *range* of R (abbreviated dom R and ran R); we recall that they are defined by

$$\text{dom } R = \{x : \textit{for some } y \ (xRy)\}$$

and

$$\text{ran } R = \{y : \textit{for some } x \ (xRy)\}.$$

If R is the relation of marriage, so that xRy menas that x is a man, y is a woman, and x and y are married to one another, then dom R is the set of married men and ran R is the set of married women.
...

SECTION 8
FUNCTIONS
If X and Y are sets, a *function* from (or *on*) X *to* (or *into*) Y is a relation f such that dom $f = X$ and such that for each x in X there is a unique element y in Y such that $(x, y) \in f$. The uniqueness condition can be formulated explicitly as follows: if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$. For each x

in X, the unique y in Y such that $(x, y) \in f$ is denoted by $f(x)$. For functions this notation and its minor variants supersede the others used for more general relations; from now on, if f is a function, we shall write $f(x) = y$ instead of $(x, y) \in f$ or xfy. The element y is called the *value* that the function f *assumes* (or *takes on*) at the *argument* x; equivalently we may say that f *sends* or *maps* or *transforms* x onto y. The words *map* or *mapping*, *transformation*, *correspondence*, and *operator* are among some of the many that are sometimes used as synonyms for *function.* The symbol

$$f : X \longrightarrow Y$$

is sometimes used as an abbreviation for "f is a function from X to Y." The set of all functions from X to Y is a subset of the power set $\mathcal{P}(X \times Y)$; it will be denoted by Y^X.

The connotations of activity suggested by synonyms listed above make some scholars dissatisfied with the definition according to which a function does not *do* anything but merely *is.* This dissatisfaction is reflected in a different use of the vocabulary: *function* is reserved for the undefined object that is somehow active, and the set of ordered pairs that we have called the function is then called the *graph* of the function. It is easy to find examples of functions in the precise set-theoretic sense of the word in both mathematics and everyday life; all we have to look for is information, not necessarily numerical, in tabulated form. One example is a city directory; the arguments of the function are, in this case, the inhabitants of the city, and their values are their addresses. ...

M. Bourbaki' *Elements of Mathematics: Theory of Sets* [1970]

4. FUNCTIONS

Definition 9. *A graph* $\mathbf{F}$ *is said to be a functional graph if for each* x *there is at most one object which corresponds to* x *under* $\mathbf{F}$ (Chapter I, §5, no. 3). *A correspondence* $f = (\mathbf{F}, \mathbf{A}, \mathbf{B})$ *is said to be a function if its graph* $\mathbf{F}$ *is a functional graph and if its source* $\mathbf{A}$ *is equal to its domain* $\mathbf{pr_1 F}$. *In other words, a correspondence* $f = (\mathbf{F}, \mathbf{A}, \mathbf{B})$ *is a function if for every* x *belonging to the source* $\mathbf{A}$ *of* f *the relation* $(x, y) \in \mathbf{F}$ *is functional in* y (Chapter I, §5, no. 3)*; the unique object which corresponds to* x *under* f *is called the value of* f *at the element* x *of* $\mathbf{A}$*, and is denoted by* $f(x)$ *(or* f_x*, or* $\mathbf{F}(x)$*, or* F_x*).*

If f is a function, $\mathbf{F}$ its graph, and x an element of the domain of f, the relation $y = f(x)$ is then equivalent to $(x, y) \in \mathbf{F}$ (Chapter I, §5, no. 3, criterion C46). ...

Let $\mathbf{A}$ and $\mathbf{B}$ be two sets; a —it mapping of $\mathbf{A}$ into $\mathbf{B}$ is a function f whose source (which is equal to its domain) is equal to $\mathbf{A}$ and whose target is equal to $\mathbf{B}$; such a function is also said to be defined on $\mathbf{A}$ and to *take its values in* $\mathbf{B}$. ...

A function f is defined on $\mathbf{A}$ is said to *transform* x *into* $f(x)$ (for all $x \in \mathbf{A}$); $f(x)$ is called the *transform of* x *by* f (by abuse of language) the *image* of x under f. ...