MATH1050 Relations, functions and ‘well-defined-ness’ for functions

1. **‘In-formal’ definition for the notion of function**.

   Recall the in-formal definition for the notion of function’:

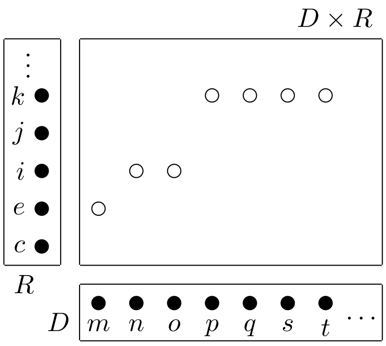
   Let D, R be sets. h is a function from D to R exactly when h is a ‘rule of assignment’ from D to R, so that

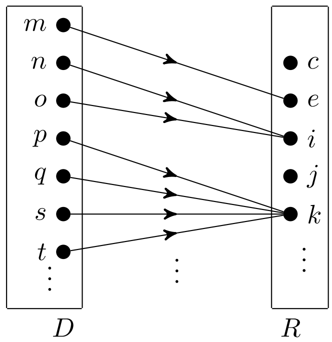
   > *each element x of D is being assigned to exactly one element, namely $h(x)$, of R.*

   D is called the domain of h. R is called the range of h.

   Below are the ‘coordinate plane diagram’ and the ‘blobs-and-arrow diagram’ for such a mathematical object, say, the function $h : D \longrightarrow R$, where:

   - $D = \{m, n, o, p, q, s, t, ...\}$, $R = \{c, e, i, j, k, ...\}$,
   - $h(m) = e, h(n) = i, h(o) = i, h(p) = k, h(q) = k, h(s) = k, h(t) = k, \cdots$, and
   - The graph of the function h is the set $H = \{(m, e), (n, i), (o, i), (p, k), (q, k), (s, k), (t, k), \cdots\}$.





2. **Problem in the ‘in-formal’ definition for the notion of function, and the solution**.

   The problem with the above ‘in-formal definition’ is that it is not clear what we mean by the phrase ‘*rule of assignment*’, which is crucial in explaining the mathematical meaning of the word ‘function’.

   To solve this problem, we try to formulate an appropriate definition for the notion of function in terms of something that we understand better mathematically: we shall write in terms of the language of sets.

   But what kind of sets shall we be looking at?

   In order to understand what any particular function does as a ‘rule of assignment’ from its domain and its range, we study its graph, which is a subset of the cartesian product of the domain and the range. In fact the graph of the function contains exactly all the information about that function: we can recover from its graph what the function does as a ‘rule of assignment’ from its domain to its range.

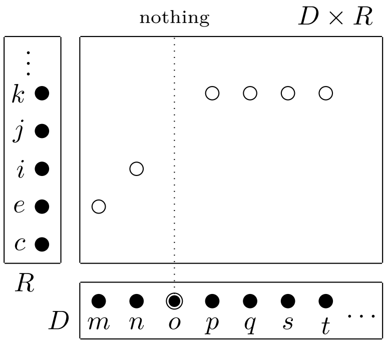
   Hence we focus on how to make sense of ‘*graph of a function*’ in terms of the language of sets, instead of making sense of ‘*rule of assignment*’.

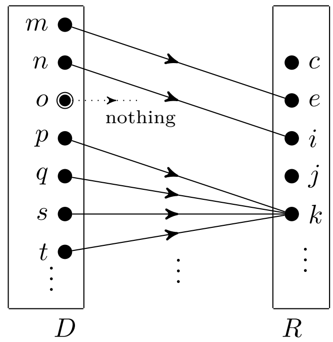
3. **Towards the formal definition for the notion of function**.

We expect the graph of a function to be necessarily a subset of the cartesian product of the domain and the range. However, it cannot be just any subset: it has to satisfy appropriate conditions so as to make sense of '*each element of the domain of the function is being assgined to exactly one element of the range of the function*', which amounts to the upholding of two kinds of forbiddance:

(a) **'First forbiddance'**:

We forbid a mathematical object from being called a function from D to R if it happens that, were it a function, its 'coordinate plane diagram' and 'blobs-and-arrows diagram' look like something as below.

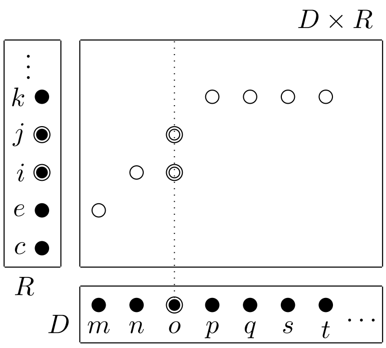


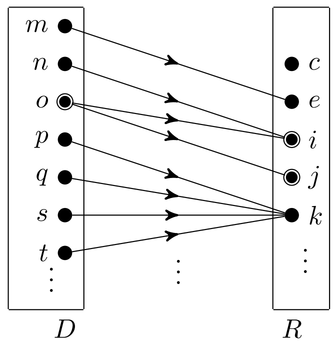


The reason for this forbiddance is that some element of D, namely o, is being assigned to no element of R.

(b) **'Second forbiddance'**:

We forbid a mathematical object from being called a function from D to R if it happens that, were it a function, its 'coordinate plane diagram' and 'blobs-and-arrows diagram' look like something as below.





The reason for this forbiddance is that some element of D, namely o, is being assigned to distinct elements of R, namely i, j.

With these two kinds of forbiddance in mind, we formulate the formal definition for the notion of functions, in terms of that for relations.

4. **Definition. (Relations.)**

Let J, K, L be sets.

The ordered triple (J, K, L) is called a **relation from J to K with graph** *L if L be a subset of $J \times K$. The sets J, K are respectively called the* **set of departure** *and the* **set of destination** *of the relation (J, K, L).*

Definition. (Functions as relations.)

Let D, R be sets, and H be a subset of $D \times R$.

The relation (D, R, H) is said to be a **function from domain D to range R with graph** *H if both of the statements $(E), (U)$ below hold:*

> (E): *For any $x \in D$, there exists some $y \in R$ such that $(x, y) \in H$.*
>
> (U): *For any $x \in D$, for any $y, z \in R$, if $(x, y) \in H$ and $(x, z) \in H$ then $y = z$.*

Where we refer to (D, R, H) as h, we may write $y = h(x)$ (or $x \underset{h}{\longmapsto} y$) exactly when $(x, y) \in H$.

Remarks.

(a) It is through the graph H of the function h that we understand how h assigns the elements of its domain D to its range R. Condition (E) and Condition (U) are formulated to describe what we want H to satisfy as a subset of $D \times R$.

(b) In plain words, When Conditions (E), (U) read:

> (E): *Each element of D is assigned by h to at least one element of R.*
>
> (U): *Each element of D is assigned by h to at most one element of R.*

So Condition (E), (U) respectively guarantee that the 'first forbiddance' and the 'second forbiddance' are upheld.

(c) The conjunction '(E) and (U)' reads:

> (EU): *Each element of D is assigned by h to exactly one element of R.*

Thus we have 'recovered' the 'in-formal definition for the notion of function'.

5. **Defining a function by making a 'declaration'.**

When we define a function, say, f, from A to B, in 'very simple' situations, we often write in this way:

- *'Define the function $f : A \longrightarrow B$ by $f(x) =$ so-and-so.'*

In such a 'declaration', we 'declare' to the reader in the *so-and-so* bit (or what we used to call the 'formula of definition' of the function f in school mathematics) how each element x of A is 'assigned' by the function f to some unique element of B, which we label $f(x)$. In fact, by doing this, we are giving a full description of all the elements of the graph of the function f, albeit not writing the description in set notations.

6. Here we have some 'very simple' situations in which we define various functions by making 'declarations'.

Examples (A).

(a) *Define the function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *by* $f(x) = x^2$ *for any* $x \in \mathbb{R}$.

This tells the reader that each real number is 'assigned' by the function f to its square, which is a uniquely determined real number.

The graph of the function f is $\{(x, x^2) \mid x \in \mathbb{R}\}$.

(b) *Define the function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *by* $f(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x > 0 \end{cases}$.

This tells the reader that each negative real number is 'assigned' by the function f to the real number -1, each positive real number to the real number 1, and the real number 0 to itself.

The graph of the function f is $\{(x, -1) \mid x < 0\} \cup \{(0, 0)\} \cup \{(y, 1) \mid y > 0\}$.

(c) *Let* $A = \{p, q, r, s, t\}$, $B = \{u, v, w, x, y, z\}$. *Define the function* $f : A \longrightarrow B$ *by* $f(p) = f(q) = w$, $f(r) = x$, $f(s) = f(t) = z$.

This tells the reader how all the elements of A, namely, p, q, r, s, t, are respectively 'assigned' by the function f to the elements of B. Each of p, q are 'assigned' to w; r to x; each of s, t to z. No element of A is 'left behind'; none is 'assigned' to something 'outside' B.

The graph of the function f is $\{(p, w), (q, w), (r, x), (s, z), (t, z)\}$.

It should be noted that this way of 'declaring' what the function we are defining 'does' to the element of its domain may lead into pit falls, if we are not careful enough. The matter concerned is usually referred to as **well-defined-ness of a function**.

7. When we define a function by making a 'declaration', we should make sure that Condition (U) is satisfied:

- Each element of the domain of the function to be defined should be assigned to at most one element of the range.

Examples (B).

Which of the below 'declarations' makes sense? Which not? Why?

(a) *Define* $f : \left\{\frac{1}{3}, \frac{2}{3}, \frac{3}{3}\right\} \longrightarrow \mathbb{R}$ *by* $f(\frac{1}{3}) = 1$, $f(\frac{2}{3}) = 2$, $f(\frac{3}{3}) = 3$.

There is no problem here: f is a function.

(b) *Define* $g : \left\{\frac{p}{q} \;\middle|\; p, q \in [\![1, 3]\!]\right\} \longrightarrow \mathbb{R}$ *by* $g(\frac{p}{q}) = p$ *for any* $p, q \in [\![1, 3]\!]$.

It does not make sense to regard g as a function, (although at first sight g looks similar to f):

- Write $A = \left\{\frac{p}{q} \;\middle|\; p, q \in [\![1, 3]\!]\right\}$.

  Note that $A = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \frac{3}{1}, \frac{3}{2}, \frac{3}{3}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, 2, \frac{2}{3}, 3, \frac{3}{2}\right\}$.

  In the above 'declaration', we require g to 'assign' the same element $\frac{1}{1}, \frac{2}{2}$ of the set A to different elements of $\mathbb{Z}$, namely $1, 2$.

(c) *Define* $h : \mathbb{Q} \longrightarrow \mathbb{R}$ *by* $h(\frac{p}{q}) = p$ *whenever* $p \in \mathbb{Z}$ *and* $q \in \mathbb{Z} \backslash \{0\}$.

It does not make sense to regard h as a function:

- In the above 'declaration', we require h to 'assign' at least one element of $\mathbb{Q}$, such as $\frac{1}{1}, \frac{2}{2}$, to different elements of $\mathbb{Z}$, namely $1, 2$.

8. When we define a function by making a 'declaration', we should make sure that Condition (E) is satisfied:

   - Each element of the domain of the function to be defined should be assigned to at least one element of the range.

   Examples (C).

   Which of the below 'declarations' makes sense? Which not? Why?

   (a) *Define* $f : \left\{\frac{1^2}{1^2}, \frac{1^2}{2^2}, \frac{2^2}{1^2}, \frac{2^2}{2^2}\right\} \longrightarrow \mathbb{R}$ *by* $f(\frac{1^2}{1^2}) = \frac{1}{1}$, $f(\frac{1^2}{2^2}) = \frac{1}{2}$, $f(\frac{2^2}{1^2}) = \frac{2}{1}$, $f(\frac{2^2}{2^2}) = \frac{2}{2}$.

   There is no problem here: f is a function.

   (b) *Define* $g : \left\{ \frac{p}{q} \;\middle|\; p, q \in [\![1, 4]\!] \right\} \longrightarrow \mathbb{R}$ *by* $g(\frac{s^2}{t^2}) = \frac{s}{t}$, *whenever* $s^2, t^2 \in [\![1, 4]\!]$ *and* $s, t \in \mathbb{N}$.

   It does not make sense to regard g as a function, (although at first sight g looks similar to f):

   - Write $A = \left\{ \frac{p}{q} \;\middle|\; p, q \in [\![1, 4]\!] \right\}$. Note that $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 2, \frac{2}{3}, 3, \frac{3}{2}, \frac{3}{4}, 4, \frac{4}{3}\right\}$.

     In the above 'declaration', we fail to require g to 'assign' some element of A, such as $\frac{2}{3}$, to some element of $\mathbb{Z}$.

   (c) *Define* $h : \mathbb{Q} \longrightarrow \mathbb{R}$ *by* $h(r^2) = r$ *whenever* $r \in \mathbb{Q}$.

   It does not make sense to regard h as a function:

   - In the above 'declaration', we fail to require h to 'assign' at least one element of $\mathbb{Q}$, such as $\frac{2}{3}$, to some element of $\mathbb{R}$.

   Examples (D).

   Which of the below 'declarations' makes sense as a 'formula of definition for a function'? Which not? Why?

   (a) *Define* $f : \mathbb{C} \longrightarrow \mathbb{R}$ *by* $f(z) = |z|$ *for any* $z \in \mathbb{C}$.

   There is no problem here: f is a function.

   (b) *Define* $g : \mathbb{C} \longrightarrow \mathbb{R}$ *by* $g(z) = i|z|$ *for any* $z \in \mathbb{C}$.

   It does not make sense to regard g as a function, (although at first sight g looks similar to f):

   - The range of the function we intend to define is $\mathbb{R}$. However, in the above 'declaration', we require g to 'assign' some element of $\mathbb{R}$, such as 1, to the number i, which is outside $\mathbb{R}$.

   (c) *Define* $h : \mathbb{C} \longrightarrow \mathbb{R}$ *by* $h(z) = \dfrac{z^4\bar{z} + iz^3(\bar{z})^2 - iz^2(\bar{z})^3 + z(\bar{z})^4}{2|z|^2 + z^2 + (\bar{z})^2 + 1}$ *for any* $z \in \mathbb{C}$.

   h is indeed **well-defined as a function** from $\mathbb{C}$ to $\mathbb{R}$. But it is not apparent until we have verified that for each $z \in \mathbb{C}$, $\dfrac{z^4\bar{z} + iz^3(\bar{z})^2 - iz^2(\bar{z})^3 + z(\bar{z})^4}{2|z|^2 + z^2 + (\bar{z})^2 + 1}$ is well-defined as a real number.

   Appropriate work should precede or follow the above 'declaration' to assure the reader that it makes sense to regard h as a function. Below is an illustration on how this is done:

   We claim that for any $z \in \mathbb{C}$, $2|z|^2 + z^2 + (\bar{z})^2 + 1 \neq 0$ *and* $\dfrac{z^4\bar{z}+iz^3(\bar{z})^2-iz^2(\bar{z})^3+z(\bar{z})^4}{2|z|^2 + z^2 + (\bar{z})^2 + 1} \in \mathbb{R}$.

   Justification: Let $z \in \mathbb{C}$. $2|z|^2 + z^2 + (\bar{z})^2 + 1 = ... \neq 0$. *Et cetera.*

   Define the function $h : \mathbb{C} \longrightarrow \mathbb{R}$ *by* $h(z) = \dfrac{z^4\bar{z} + iz^3(\bar{z})^2 - iz^2(\bar{z})^3 + z(\bar{z})^4}{2|z|^2 + z^2 + (\bar{z})^2 + 1}$ *for any* $z \in \mathbb{C}$.

   So, by the time we write down what h 'does', we have already made sure that no element of $\mathbb{C}$ is 'assigned' by h to something outside the intended range of h.

9. It may happen that when we define a function by making a ‘declaration’, it is neither immediately clear that Condition (E) is satisfied nor immediately clear that Condition (U) is satisfied.

In this situation, rather than ‘declaring’ through a ‘formula of definition’ what the function to be defined ‘does’, it may be better to first express explicitly in set notations the graph of the function to be defined and then verify that Condition (E) and Condition (U) are satisfied.

Examples (E).

Write $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Here we wonder whether the below ‘declaration’ makes sense:

- *Define* $g : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ *by* $g(z) = \sqrt{|z|}(\cos(2\theta) + i\sin(2\theta))$ *whenever* $z \in \mathbb{C}^*$ *and* θ *is an argument of* z.

Note that if g is a function then its graph is given by the set

$$\left\{ (z,\zeta) \;\middle|\; \begin{array}{l} z,\zeta \in \mathbb{C}^* \text{ and there exists some } \theta \in \mathbb{R} \text{ such that} \\ z = |z|(\cos(\theta) + i\sin(\theta)) \text{ and } \zeta = \sqrt{|z|}(\cos(2\theta) + i\sin(2\theta)) \end{array} \right\}.$$

We proceed to check that g is well-defined as a function below:

Define the subset G *of* $\mathbb{C}^* \times \mathbb{C}^*$ *by*

$$G = \left\{ (z,\zeta) \;\middle|\; \begin{array}{l} z,\zeta \in \mathbb{C}^* \textit{ and there exists some } \theta \in \mathbb{R} \textit{ such that} \\ z = |z|(\cos(\theta) + i\sin(\theta)) \textit{ and } \zeta = \sqrt{|z|}(\cos(2\theta) + i\sin(2\theta)) \end{array} \right\}.$$

Define $g = (\mathbb{C}^*, \mathbb{C}^*, G)$.

∗ *[Does g satisfy Condition (E)?]*
Pick any $z \in \mathbb{C}^*$. *There exists some* $\theta \in \mathbb{R}$ *such that* $z = |z|(\cos(\theta) + i\sin(\theta))$.
Define $\zeta = \sqrt{|z|}(\cos(2\theta) + i\sin(2\theta))$.
Since $z \in \mathbb{C}^*$, $\sqrt{|z|} \neq 0$. *Also note that* $|\cos(2\theta) + i\sin(2\theta)| = 1$. *Then* $\zeta \in \mathbb{C}^*$.
By definition, we have $(z,\zeta) \in G$.

∗ *[Does g satisfy Condition (U)?]*
Pick any $z \in \mathbb{C}^*$. *Pick any* $\zeta, \omega \in \mathbb{C}^*$. *Suppose* $(z,\zeta) \in G$ *and* $(z,\omega) \in G$.
Since $(z,\zeta) \in G$, *there exist some* $\theta \in \mathbb{R}$ *such that* $z = |z|(\cos(\theta)+i\sin(\theta))$ *and* $\zeta = \sqrt{|z|}(\cos(2\theta)+i\sin(2\theta))$.
Since $(z,\omega) \in G$, *there exist some* $\varphi \in \mathbb{R}$ *such that* $z = |z|(\cos(\varphi)+i\sin(\varphi))$ *and* $\omega = \sqrt{|z|}(\cos(2\varphi)+i\sin(2\varphi))$.
Note that $|z|(\cos(\theta) + i\sin(\theta)) = z = |z|(\cos(\varphi) + i\sin(\varphi))$. *Since* $|z| \neq 0$, *we have* $\cos(\theta) = \cos(\varphi)$ *and* $\sin(\theta) = \sin(\varphi)$.
Then there exists some $k \in \mathbb{Z}$ *such that* $\theta = \varphi + 2k\pi$. *Therefore*

$$\begin{aligned} \zeta = \sqrt{|z|}(\cos(2\theta) + i\sin(2\theta)) &= \sqrt{|z|}(\cos(2(\theta + 2k\pi)) + i\sin(2(\theta + 2k\pi))) \\ &= \sqrt{|z|}(\cos(2\varphi) + i\sin(2\varphi)) = \omega. \end{aligned}$$

It follows that $g : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ *is a function, whose ‘formula of definition’ is given by* $g(z) = \sqrt{|z|}(\cos(2\theta)+i\sin(2\theta))$ *whenever* $z \in \mathbb{C}^*$ *and* θ *is an argument of* z.