1. **'In-formal' definition for the notion of function**.

Recall the in-formal definition for the notion of function':

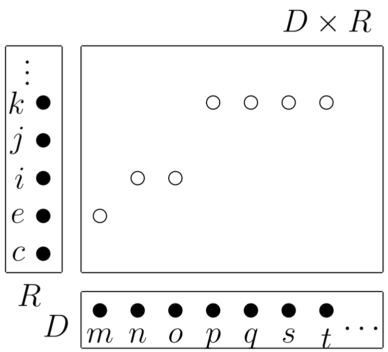
Let D, R be sets.

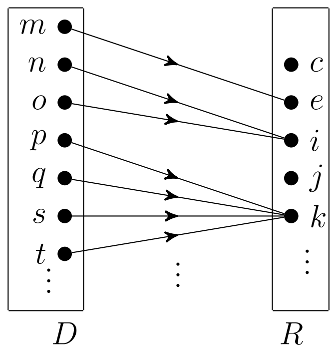
h is a function from D to R exactly when h is a 'rule of assignment' from D to R, so that

each element x of D is being assigned to exactly one element, namely $h(x)$, of R.

D is called the domain of h. R is called the range of h.

Below are the 'coordinate plane diagram' and the 'blobs-and-arrow diagram' for such a mathematical object, say, the function $h : D \longrightarrow R$.





Here $D = \{m, n, o, p, q, s, t, ...\}$, $R = \{c, e, i, j, k, ...\}$,
$h(m) = e, h(n) = i, h(o) = i, h(p) = k, h(q) = k, h(s) = k, h(t) = k, \cdots$, and
the graph of h is the set $H = \{(m, e), (n, i), (o, i), (p, k), (q, k), (s, k), (t, k), \cdots\}$.

2. **Problem in the 'in-formal' definition for the notion of function, and the solution.**

- What do we mean by the phrase '*rule of assignment*'?

The meaning of this phrase is unclear.
But this phrase is crucial in explaining the mathematical meaning of the word 'function'.

- How to solve this problem?

Formulate an appropriate definition for the notion of function in terms of something that we understand better mathematically: language of sets.

- But what kind of sets shall we be looking at? Why?

The graph of a function is a subset of the cartesian product of the domain and the range of the function.
This carries all the information of the function as a 'rule of assignment' from its domain to its range.
So we focus on making sense of the phrase 'graph of a function'.

3. **Towards the formal definition for the notion of function**.

We expect the graph of a function to be necessarily a subset of the cartesian product of the domain and the range.

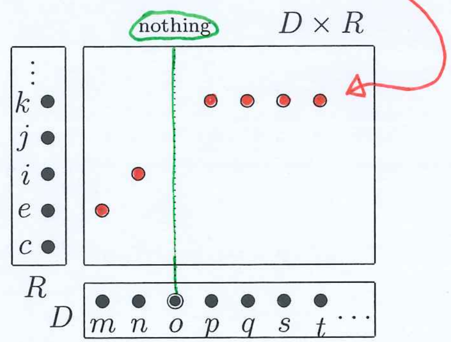
However, it cannot be just any subset:

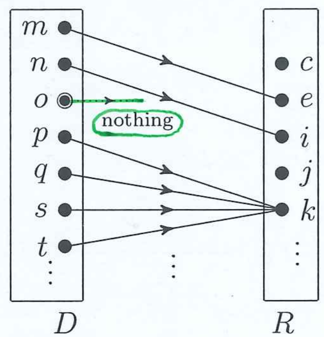
As a 'rule of assignment', the function has to satisfy:

each element of its domain is being assigned to exactly one element of its range.

(a) **'First forbiddance'**:

Subsets of $D \times R$ like the one below will not be allowed to be the graph of any function:





The reason for this forbiddance is that some element of D, namely o, is being assigned to no element of R.

To be more precise?

[Write $G = \{(m,e),(n,i),(p,k),(q,k),(s,k),(t,k),\dots\}$.

For this G, there exists some $x \in D$, namely $x = o$, such that for any $y \in R$, $(x,y) \notin G$.]

Towards the formal definition for the notion of function.

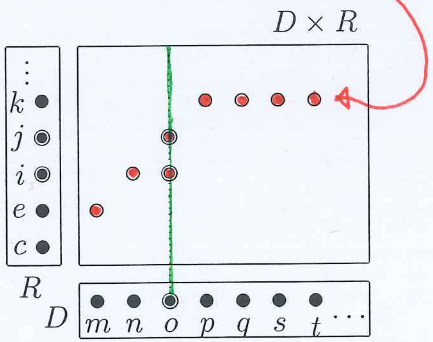
We expect the graph of a function to be necessarily a subset of the cartesian product of the domain and the range.

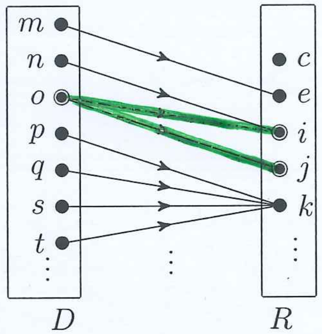
However, it cannot be just any subset:
As a 'rule of assignment', the function has to satisfy:
each element of its domain is being assigned to exactly one element of its range.

(a) 'First forbiddance': ...

(b) **'Second forbiddance'**:

Subsets of $D \times R$ like the one below will not be allowed to be the graph of any function:





The reason for this forbiddance is that some element of D, namely o, is being assigned to distinct elements of R, namely i, j.

To be more precise?

[Write $G = \{(m,e), (n,i), (o,i), (o,j), (p,k), (q,k), (s,k), (t,k), \ldots\}$
For this G, there exists some $x \in D$, $y, z \in R$, namely $x = o$, $y = i$, $z = j$, such that $(x,y) \in G$ and $(x,z) \in G$ and $y \neq z$.]

4. **Definition. (Relations.)**

Let J, K, L be sets.

The ordered triple (J, K, L) is called a **relation from J to K with graph** *L if L be a subset of $J \times K$.*

The sets J, K are respectively called the **set of departure** *and the* **set of destination** *of the relation (J, K, L).*

Definition. (Functions as relations.)

Let D, R be sets, and H be a subset of $D \times R$.

The relation (D, R, H) is said to be a **function from domain D to range R with graph** *H if both of the statements $(E), (U)$ below hold:*

(E): For any $x \in D$, there exists some $y \in R$ such that $(x, y) \in H$.

(U): For any $x \in D$, for any $y, z \in R$, if $(x, y) \in H$ and $(x, z) \in H$ then $y = z$.

Where we refer to (D, R, H) as h, we may write $y = h(x)$ (or $x \underset{h}{\longmapsto} y$) exactly when $(x, y) \in H$.

Definition. (Functions as relations.)

Let D, R be sets, and H be a subset of $D \times R$.

The relation (D, R, H) is said to be a **function from domain D to range R with graph H** *if both of the statements $(E), (U)$ below hold:*

(E): *For any $x \in D$, there exists some $y \in R$ such that $(x, y) \in H$.*

(U): *For any $x \in D$, for any $y, z \in R$, if $(x, y) \in H$ and $(x, z) \in H$ then $y = z$.*

Remarks.

(a) It is through the graph H of the function h that we understand how h assigns the elements of its domain D to its range R.

Condition (E) and Condition (U) are formulated to describe what we want H to satisfy as a subset of $D \times R$.

(b) In plain words, When Conditions (E), (U) read:

(E): *Each element of D is assigned by h to at least one element of R.*

(U): *Each element of D is assigned by h to at most one element of R.*

So Condition (E), (U) respectively guarantee that the 'first forbiddance' and the 'second forbiddance' are upheld.

(c) The conjunction '(E) and (U)' reads:

(EU): *Each element of D is assigned by h to exactly one element of R.*

Thus we have 'recovered' the 'in-formal definition for the notion of function'.

5. **Defining a function by making a 'declaration'.**

When we define a function, say, f, from A to B, in 'very simple' situations, we often write in this way:

- '*Define the function* $f : A \longrightarrow B$ *by* $f(x) =$ so-and-so.'

What is being done by such a declaration?

We are giving a full description of all the elements of the graph of the function f, albeit not writing the description in set notations.

6. **Examples (A).**

(a) *Define the function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *by* $f(x) = x^2$ *for any* $x \in \mathbb{R}$.

Graph of the function f? $\{(x, x^2) \mid x \in \mathbb{R}\}$

(b) *Define the function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *by* $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$.

Graph of the function f? $\{(s,-1) \mid s<0\} \cup \{(0,0)\} \cup \{(t,1) \mid t>0\}$

(c) *Let* $A = \{p, q, r, s, t\}$, $B = \{u, v, w, x, y, z\}$. *Define the function* $f : A \longrightarrow B$ *by* $f(p) = f(q) = w$, $f(r) = x$, $f(s) = f(t) = z$.

Whenever we define a function in this way, we have to be careful with **well-defined-ness of a function**.

7. **Examples (B)**.

Which of the below 'declarations' makes sense? Which not? Why?

(a) *Define* $f : \left\{\frac{1}{3}, \frac{2}{3}, \frac{3}{3}\right\} \longrightarrow \mathbb{R}$ *by* $f(\frac{1}{3}) = 1$, $f(\frac{2}{3}) = 2$, $f(\frac{3}{3}) = 3$.

No problem. f is a function.

(b) *Define* $g : \left\{ \frac{p}{q} \,\middle|\, p, q \in [\![1, 3]\!] \right\} \longrightarrow \mathbb{R}$ *by* $g(\frac{p}{q}) = p$ *for any* $p, q \in [\![1, 3]\!]$.

It does not make sense to regard g as a function.

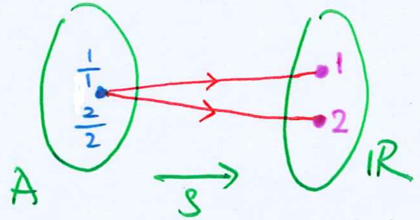
- Write $A = \{ \frac{p}{q} \mid p, q \in [\![1, 3]\!] \}$.

$A = \{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \frac{3}{1}, \frac{3}{2}, \frac{3}{3} \} = \{ 1, \frac{1}{2}, \frac{1}{3}, 2, \frac{2}{3}, 3, \frac{3}{2} \}$.

g assigns $\frac{1}{1}$ to 1.

g assigns $\frac{2}{2}$ to 2.

But $\frac{1}{1} = \frac{2}{2} = 1$.



(c) *Define* $h : \mathbb{Q} \longrightarrow \mathbb{R}$ *by* $h(\frac{p}{q}) = p$ *whenever* $p \in \mathbb{Z}$ *and* $q \in \mathbb{Z} \backslash \{0\}$.

It does not make sense to regard h as a function.

h assigns $\frac{1}{1}$ to 1. h assigns $\frac{2}{2}$ to 2. But $\frac{1}{1} = \frac{2}{2} = 1$.

8. **Examples (C).**

Which of the below 'declarations' makes sense? Which not? Why?

(a) *Define* $f : \left\{\frac{1^2}{1^2}, \frac{1^2}{2^2}, \frac{2^2}{1^2}, \frac{2^2}{2^2}\right\} \longrightarrow \mathbb{R}$ *by* $f(\frac{1^2}{1^2}) = \frac{1}{1}$, $f(\frac{1^2}{2^2}) = \frac{1}{2}$, $f(\frac{2^2}{1^2}) = \frac{2}{1}$, $f(\frac{2^2}{2^2}) = \frac{2}{2}$.

No problem. f is a function.

(b) *Define* $g : \left\{ \frac{p}{q} \,\middle|\, p, q \in [\![1, 4]\!] \right\} \longrightarrow \mathbb{R}$ *by* $g(\frac{s^2}{t^2}) = \frac{s}{t}$, *whenever* $s^2, t^2 \in [\![1, 4]\!]$ and $s, t \in \mathbb{N}$.

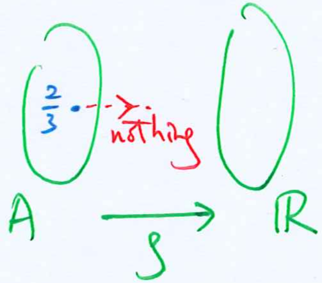
It does not make sense to regard g as a function.

- Write $A = \left\{ \frac{p}{q} \,\middle|\, p, q \in [\![1, 4]\!] \right\}$.

  $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 2, \frac{2}{3}, 3, \frac{3}{2}, \frac{3}{4}, 4, \frac{4}{3}\right\}$.

  For any $s, t \in \mathbb{N}$, if $s^2, t^2 \in [\![1, 4]\!]$ then $\frac{s^2}{t^2} \neq \frac{2}{3}$.

  g does not assign $\frac{2}{3}$ to any element of $\mathbb{R}$.



(c) *Define* $h : \mathbb{Q} \longrightarrow \mathbb{R}$ *by* $h(r^2) = r$ *whenever* $r \in \mathbb{Q}$.

It does not make sense to regard h as a function.

$\frac{2}{3} \in \mathbb{Q}$. But, for any $r \in \mathbb{Q}$, $r^2 \neq \frac{2}{3}$.

h does not assign $\frac{2}{3}$ to any element of $\mathbb{R}$.

Examples (D).

Which of the below 'declarations' makes sense as a 'formula of definition for a function'? Which not? Why?

(a) *Define* $f : \mathbb{C} \longrightarrow \mathbb{R}$ *by* $f(z) = |z|$ *for any* $z \in \mathbb{C}$.

No problem. f is a function.

(b) *Define* $g : \mathbb{C} \longrightarrow \mathbb{R}$ *by* $g(z) = i|z|$ *for any* $z \in \mathbb{C}$.

It does not make sense to regard g as a function.

$1 \in \mathbb{C}$. $i \cdot |1| = i \notin \mathbb{R}$.

g assigns $1 \in \mathbb{C}$ to something outside $\mathbb{R}$, namely i.

(c) *Define* $h : \mathbb{C} \longrightarrow \mathbb{R}$ *by* $h(z) = \dfrac{z^4\bar{z} + iz^3(\bar{z})^2 - iz^2(\bar{z})^3 + z(\bar{z})^4}{2|z|^2 + z^2 + (\bar{z})^2 + 1}$ *for any* $z \in \mathbb{C}$.

h is well-defined as a function, but it has to be justified:

- Claim: For any $z \in \mathbb{C}$, $2|z|^2 + z^2 + (\bar{z})^2 + 1 \neq 0$.
- Further claim: For any $z \in \mathbb{C}$, $\dfrac{z^4\bar{z} + iz^3(\bar{z})^2 - iz^2(\bar{z})^3 + z(\bar{z})^4}{2|z|^2 + z^2 + (\bar{z})^2 + 1} \in \mathbb{R}$.

9. **Examples (E).**

Write $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Here we wonder whether the below 'declaration' makes sense:

- *Define* $g : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ *by* $g(z) = \sqrt{|z|}(\cos(2\theta) + i\sin(2\theta))$ *whenever* $z \in \mathbb{C}^*$ *and* θ *is an argument of* z.

Note that if g is a function then its graph is given by the set

$$\left\{ (z, \zeta) \;\middle|\; \begin{array}{l} z, \zeta \in \mathbb{C}^* \text{ and there exists some } \theta \in \mathbb{R} \text{ such that} \\ z = |z|(\cos(\theta) + i\sin(\theta)) \text{ and } \zeta = \sqrt{|z|}(\cos(2\theta) + i\sin(2\theta)) \end{array} \right\}.$$

We proceed to check that g is well-defined as a function below:

Define the subset G *of* $\mathbb{C}^* \times \mathbb{C}^*$ *by*

$$G = \left\{ (z, \zeta) \;\middle|\; \begin{array}{l} z, \zeta \in \mathbb{C}^* \textit{ and there exists some } \theta \in \mathbb{R} \textit{ such that} \\ z = |z|(\cos(\theta) + i\sin(\theta)) \textit{ and } \zeta = \sqrt{|z|}(\cos(2\theta) + i\sin(2\theta)) \end{array} \right\}.$$

Define $g = (\mathbb{C}^*, \mathbb{C}^*, G)$.

$*$ *[Does g satisfy Condition (E)?]*

$*$ *[Does g satisfy Condition (U)?]*