MATH1050 Examples on finding inverse functions for 'simple' bijective functions.

0. Recall the definition for the notion of inverse functions:

   Let A, B be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ be functions. g is said to be an **inverse function** *of f if both of the following statements hold:*

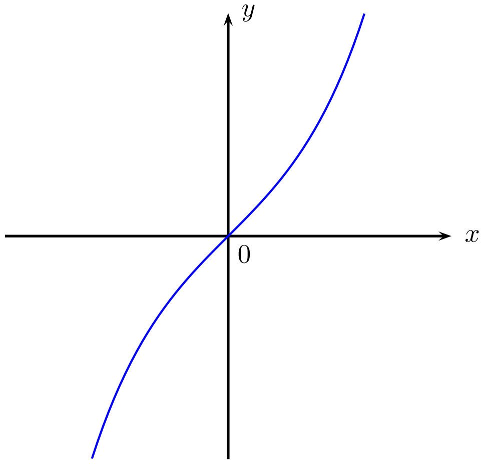
   (a) *For any $x \in A$, $(g \circ f)(x) = x$.*

   (b) *For any $y \in B$, $(f \circ g)(y) = y$.*

   In each of the examples below, we are going to determine, for the given function, whether it has an inverse function, and how to find it if it does.

1. **Example (1).**

   Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \sinh(x)$ for any $x \in \mathbb{R}$.



   We ask:

   (1) Does f have an inverse function?

   (2) If yes, how to determine (the formula of definition) for the inverse function of f explicitly?

   (3) How are the coordinate plane diagrams for f and its inverse function (if it exists) related?

   Note that f is bijective. (Check surjectivity and injectivity as exercises.) If we take for granted the logical equivalence of bijectivity and existence of inverse function, we will expect that the answer to Question (1) is yes.

   Here, however, we choose another approach to these questions. We start with Question (2), obtaining a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ which will be the inverse function of f if such exists. Then we will come back to Question (1), to see whether such a function g is indeed the inverse function of f, with direct reference to the definition for the notion of inverse function. We will consider Question (3) last.

   (a) Suppose f has an inverse function g. How to determine the 'formula of definition' of the function g?

   Answer. According to definition, for such a function g, it happens that:

   for any $x, y \in \mathbb{R}$, $y = f(x)$ iff $x = g(y)$.

   Wit this in mind, we attempt to 'change subject to x' in the 'relation' '$y = f(x)$' to obtain the 'relation' '$x = g(y)$'. Below is the calculation, to be done in practice:

   Let $x, y \in \mathbb{R}$. Suppose $y = f(x)$. Then $(e^x - y)^2 = (e^x)^2 - 2ye^x + y^2 = y^2 + 1$.
   We have $e^x = y \pm \sqrt{y^2 + 1}$.
   '$e^x = y - \sqrt{y^2 + 1}$' is rejected. (Why?)
   Therefore $x = \ln(y + \sqrt{y^2 + 1})$.

   It follows that $g(y) = \ln(y + \sqrt{y^2 + 1})$ for any $y \in \mathbb{R}$.

   Remark. What roles do surjectivity and injectivity play in the above calculation?

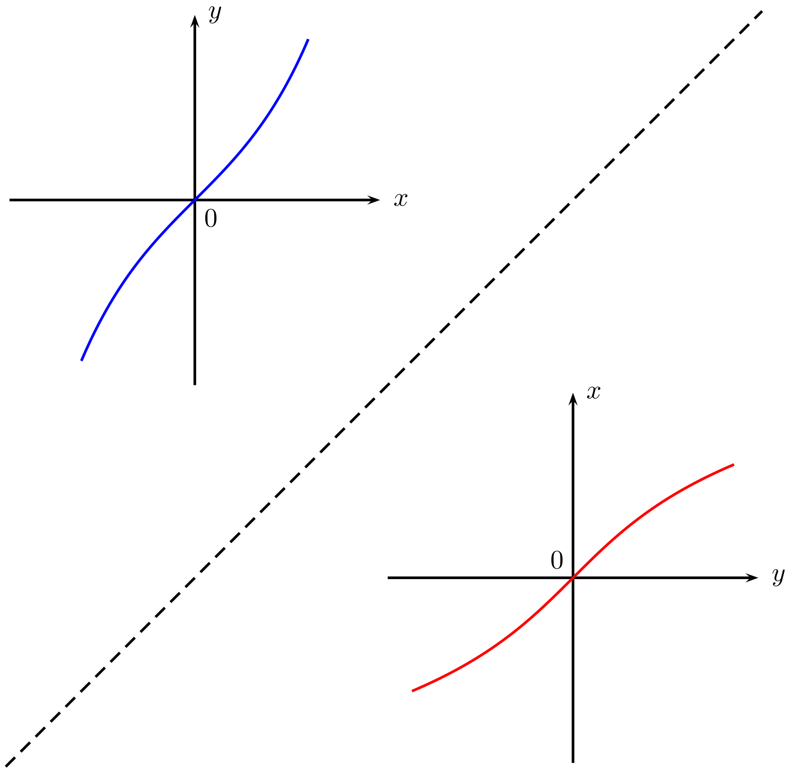
   - Surjectivity of f guarantees that for each $y \in \mathbb{R}$, the equation $y = f(u)$ with unknown u has at least one solution in $\mathbb{R}$. So each $y \in \mathbb{R}$ 'has a chance' to be 'assigned' to some $x \in \mathbb{R}$ via the 'relation' $y = f(x)$.
   - Injectivity of f guarantees that for each $y \in \mathbb{R}$, the equation $y = f(u)$ with unknown u has at most one solution in $\mathbb{R}$. So each $y \in \mathbb{R}$ will not be 'assigned' to more than one $x \in \mathbb{R}$ via the 'relation' $y = f(x)$.

   (b) Does f indeed have an inverse function?

   Answer. Check that the function $g : \mathbb{R} \longrightarrow \mathbb{R}$, given by $g(y) = \ln(y + \sqrt{y^2 + 1})$ for any $y \in \mathbb{R}$, which is found above, is indeed an inverse function of f:

- For each $x \in \mathbb{R}$, $(g \circ f)(x) = g(f(x)) = \ln(\sinh(x) + \sqrt{\sinh^2(x) + 1}) = ... = x$. (Fill in the detail.)
- For each $y \in \mathbb{R}$, $(f \circ g)(y) = f(g(y)) = \sinh(\ln(y + \sqrt{y^2 + 1})) = ... = y$. (Fill in the detail.)

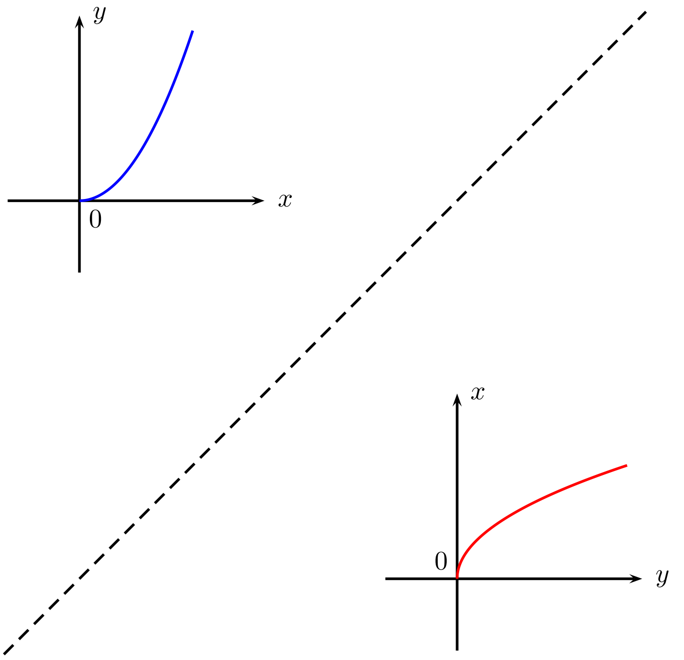
(c) What is the graph of g? How is it related to the graph of f? How to obtain it from the graph of f?
Answer. 'Flip' the picture below along a line parallel to the line $y = x$:



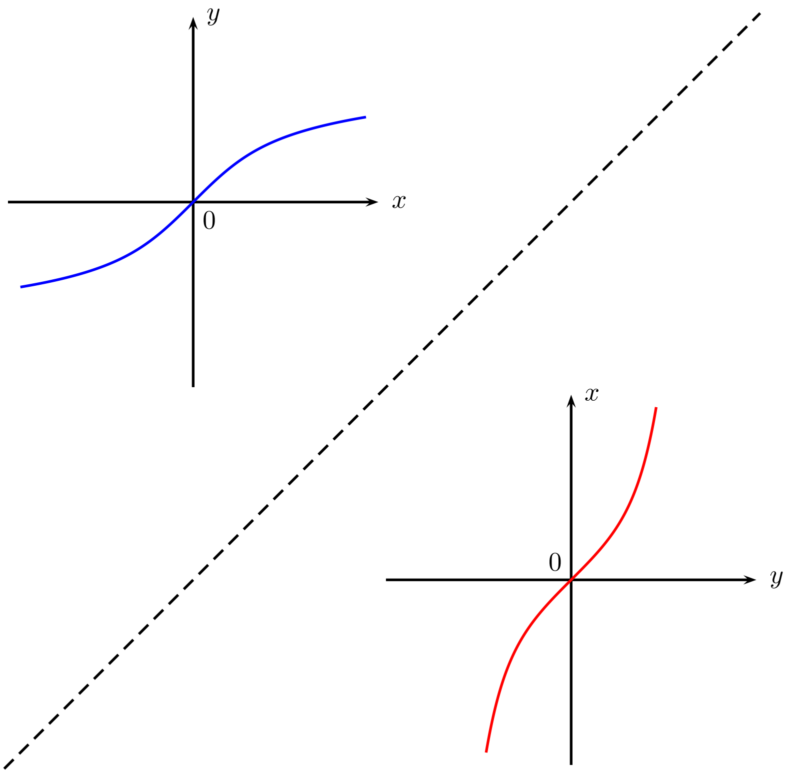
2. **Example (2): Finding inverse functions for 'simple' real-valued functions of one real variable.**

Ideas used in Example (1) can be used in similarly 'simple' examples.

(a) Let n be a positive integer, and $f : [0, +\infty) \longrightarrow [0, +\infty)$ be the function defined by $f(x) = x^n$ for any $x \in [0, +\infty)$.
How to find the inverse function of f, if it exists?
Answer. We can 'solve for' the inverse function of f explicitly. It is the function $g : [0, +\infty) \longrightarrow [0, +\infty)$, given by $g(y) = y^{1/n}$ for any $y \in [0, +\infty)$.
How to obtain the graph of the inverse function of f?
Answer. 'Flip' the picture below along a line parallel to the line $y = x$:



(b) Let $f : \mathbb{R} \longrightarrow (-1, 1)$ be the function defined by $f(x) = \tanh(x)$ for any $x \in \mathbb{R}$.
How to find the inverse function of f, if it exists?
Answer. We can 'solve for' the inverse function of f explicitly. It is $g : (-1, 1) \longrightarrow \mathbb{R}$, given by $g(y) = \frac{1}{2} \ln\left(\frac{1+y}{1-y}\right)$ for any $y \in (-1, 1)$.
How to obtain the graph of the inverse function of f?
Answer. 'Flip' the picture below along a line parallel to the line $y = x$:



3. **Example (3): Finding inverse functions for 'simple' bijective complex-valued functions of one complex variable.**

The same 'algebraic' ideas in Example (1), Example (2) also apply when 'real' is changed to 'complex', '$\mathbb{R}$' to '$\mathbb{C}$', et cetera.

(a) Let $a, b \in \mathbb{C}$. Suppose $a \neq 0$.
Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = az + b$ for any $z \in \mathbb{C}$.
What is the inverse function of f, if it exists? How to find it?
Answer. 'Solve for it'; perform 'change-of-subject'.
It is $g : \mathbb{C} \longrightarrow \mathbb{C}$ given by

$$g(w) = \frac{w - b}{a} \qquad \text{for any } w \in \mathbb{C}.$$

(b) Let $a, b \in \mathbb{C}$. Suppose $a \neq 0$.
Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = a\bar{z} + b$ for any $z \in \mathbb{C}$.
What is the inverse function of f, if it exists? How to find it?
Answer. 'Solve for it'; perform 'change-of-subject'.
It is $g : \mathbb{C} \longrightarrow \mathbb{C}$ given by

$$g(w) = \frac{\bar{w} - b}{a} \qquad \text{for any } w \in \mathbb{C}.$$

(c) Let $f : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{C}\backslash\{0\}$ be the function defined by $f(z) = \dfrac{1}{z}$ for any $z \in \mathbb{C}\backslash\{0\}$.
What is the inverse function of f, if it exists? How to find it?
Answer. 'Solve for it'; perform 'change-of-subject'.
It is $g : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{C}\backslash\{0\}$ given by

$$g(w) = \frac{1}{w} \qquad \text{for any } w \in \mathbb{C}\backslash\{0\}.$$

(d) Let $a, b, c, d \in \mathbb{C}$. Suppose $c \neq 0$ and $ad - bc \neq 0$.
Let $f : \mathbb{C}\backslash\{-d/c\} \longrightarrow \mathbb{C}\backslash\{a/c\}$ be the function defined by $f(z) = \dfrac{az + b}{cz + d}$ for any $z \in \mathbb{C}\backslash\{-d/c\}$.
What is the inverse function of f, if it exists? How to find it?
Answer. 'Solve for it'; perform 'change-of-subject'.
It is $g : \mathbb{C}\backslash\{a/c\} \longrightarrow \mathbb{C}\backslash\{-d/c\}$ given by

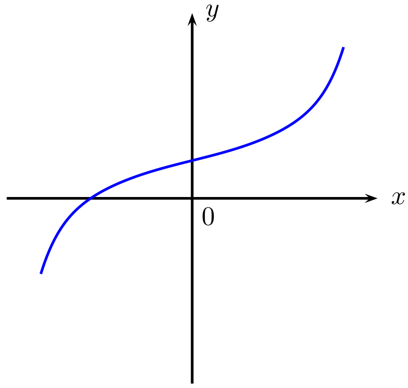
$$g(w) = \frac{dw - b}{-cw + a} \qquad \text{for any } w \in \mathbb{C}\backslash\{a/c\}.$$

(e) Let $H = \{z \in \mathbb{C} : \mathsf{Re}(z) > 0\}$, and $N = \{w \in \mathbb{C} : w \in \mathbb{R} \text{ and } w \leq 0\}$.
Let $f : H \longrightarrow \mathbb{C}\backslash N$ be the function defined by $f(z) = z^2$ for any $z \in H$.
What is the inverse function of f, if it exists? How to find it?
Answer. 'Solve for it'; perform 'change-of-subject'. (Quadratic equations and surd forms are involved. Be careful with signs.)
It is $g : \mathbb{C}\backslash N \longrightarrow H$ given by

$$g(w) = \sqrt{\frac{|w| + \mathsf{Re}(w)}{2}} + \frac{i\mathsf{Im}(w)}{\sqrt{2} \cdot \sqrt{|w| + \mathsf{Re}(w)}} \qquad \text{for any } w \in \mathbb{C}\backslash N.$$

4. **Example (4): what if 'school algebra' does not work?**

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{x^9}{1024} + \dfrac{x^3}{16} + \dfrac{x}{4} + \dfrac{1}{2}$ for any $x \in \mathbb{R}$.



We ask:

(1) Does f have an inverse function?

(2) If yes, how to determine (the formula of definition) for the inverse function of f explicitly?

(3) How are the coordinate plane diagrams for f and its inverse function (if it exists) related?

With the help of the Intermediate Value Theorem, we can check that f is bijective. (For detail, see the Handout *Intermediate Value Theorem, and the surjectivity and injectivity for continuous real-valued functions of one real-variable.*) If we take for granted the logical equivalence of bijectivity and existence of inverse function, we will expect that the answer to Question (1) is yes.

As for question (2), it will transpire that there is no chance for us to 'solve for it' using 'school maths algebra'. (You will learn the reason in a course on *Galois Theory.*)

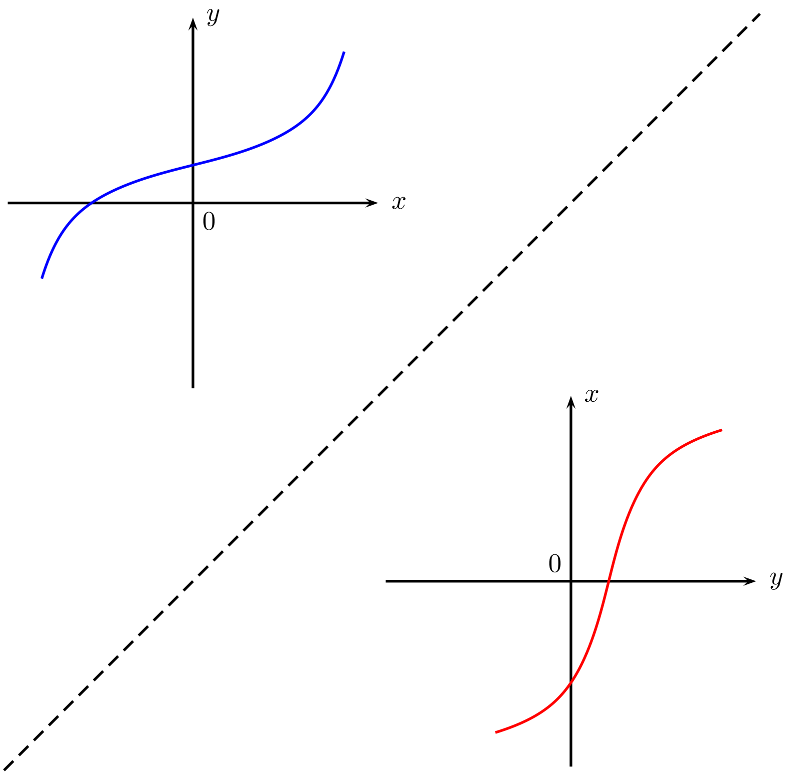
However, if we do not insist on writing down the 'explicit formula', we can give a full description of the inverse function of f, through the answer to Question (3).

According to definition, a necessary and sufficient condition for a function, say, $g : \mathbb{R} \longrightarrow \mathbb{R}$, to be an inverse function of f is that

- 'for any $x, y \in \mathbb{R}$, $y = f(x)$ iff $x = g(y)$.'

So we will expect the graph of such a function g to be given by the set $\left\{ (y, x) \;\middle|\; y, x \in \mathbb{R} \text{ and } y = \dfrac{x^9}{1024} + \dfrac{x^3}{16} + \dfrac{x}{4} + \dfrac{1}{2} \right\}$.

We can visualize the function g when we 'flip' the picture for the coordinate plane diagram for function f below along a line parallel to the line $y = x$.



5. **Example (5): What is arctangent?**

Let $f : \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \tan(x)$ for any $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

How to find the inverse function of f, if it exists?

Answer. With the help of the Intermediate Value Theorem, we can check that f is bijective.

So we expect that f has an inverse function. It is $g : \mathbb{R} \longrightarrow \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, given by $g(y) = \arctan(y)$ for any $y \in \mathbb{R}$. (We are used to calling it the **arctangent function**.)

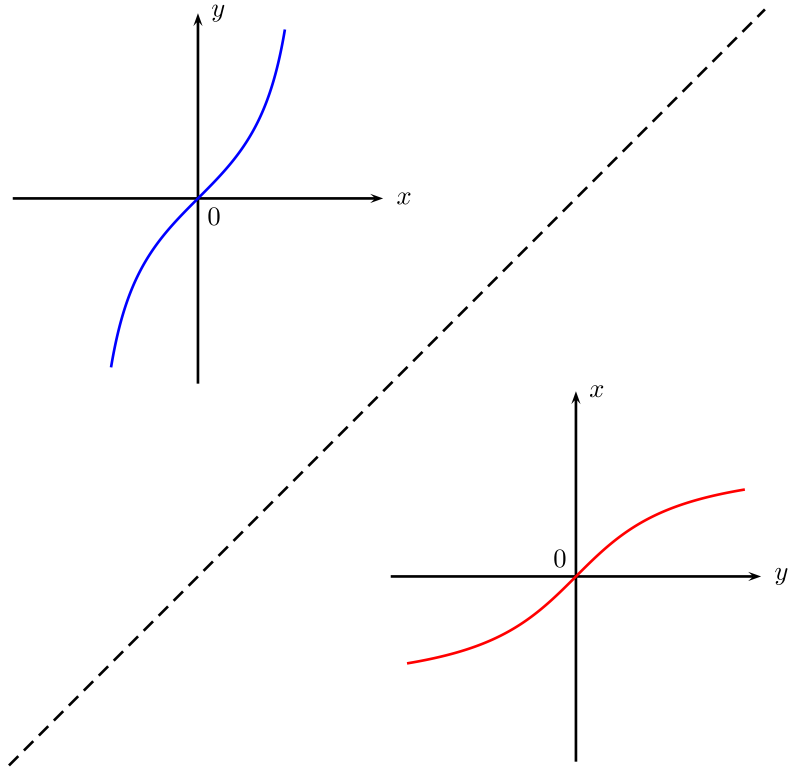
With the help of the Inverse Function Theorem and the Fundamental Theorem of the Calculus, we find that

$$g(y) = \int_0^y \frac{dt}{1+t^2} \qquad \text{for any } y \in \mathbb{R}.$$

But what is g really?

Answer. g is the function whose graph is the set $\left\{ (y, x) \;\middle|\; y \in \mathbb{R} \text{ and } x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \text{ and } y = \tan(x) \right\}$.

g is obtained when we 'flip' the picture for the coordinate plane diagram for function f below along a line parallel to the line $y = x$.



6. **Example (6): What is arcsine?**

Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow (-1, 1)$ be the function defined by $f(x) = \sin(x)$ for any $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

How to find the inverse function of f, if it exists?

Answer. With the help of the Intermediate Value Theorem, we can check that f is bijective.

So we expect that f has an inverse function. It is $g : (-1, 1) \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, given by $g(y) = \arcsin(y)$ for any $y \in (-1, 1)$. (We are used to calling it the **arcsine function**.)

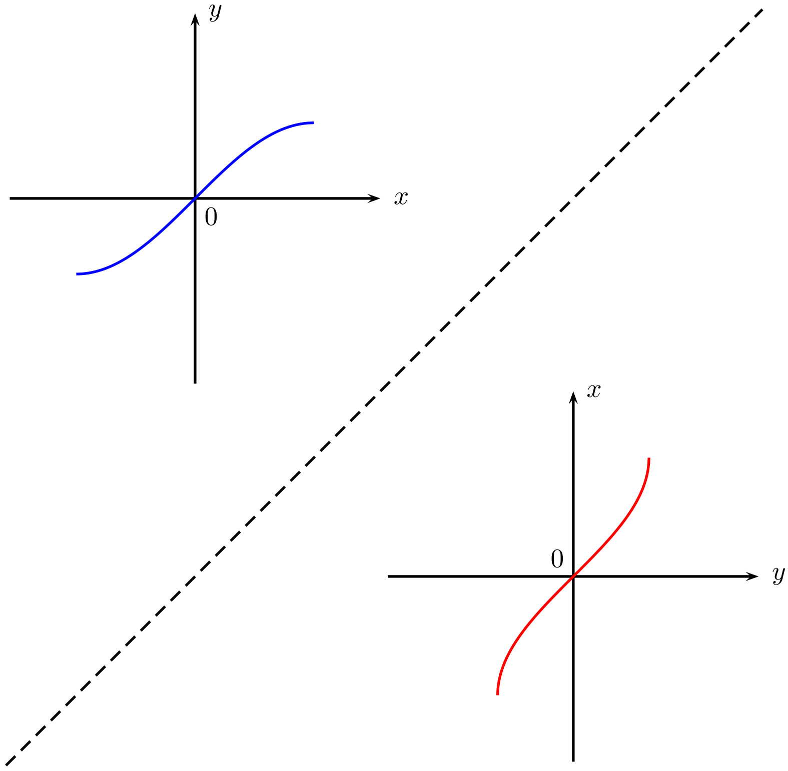
With the help of the Inverse Function Theorem and the Fundamental Theorem of the Calculus, we find that

$$g(y) = \int_0^y \frac{dt}{\sqrt{1-t^2}} \qquad \text{for any } y \in (-1, 1).$$

But what is g really?

Answer. g is the function whose graph is the set $\left\{(y, x) \;\middle|\; y \in (-1, 1) \text{ and } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ and } y = \sin(x)\right\}$.

g is obtained when we 'flip' the picture for the coordinate plane diagram for function f below along a line parallel to the line $y = x$.



7. **Example (7): What is natural logarithm?**

   Let $f : \mathbb{R} \longrightarrow (0, +\infty)$ be the function defined by $f(x) = \exp(x)$ for any $x \in \mathbb{R}$.

   How to find the inverse function of f, if it exists?

   Answer. With the help of the Intermediate Value Theorem, we can check that f is bijective.

   So we expect that f has an inverse function. It is $g : (0, +\infty) \longrightarrow \mathbb{R}$ given by $g(y) = \ln(y)$ for any $y \in (0, +\infty)$. (We are used to calling it the **natural logarithmic function**.)

   With the help of the Inverse Function Theorem and the Fundamental Theorem of the Calculus, we find that

$$g(y) = \int_1^y \frac{dt}{t} \quad \text{for any } y \in (0, +\infty).$$

   But what is g really?

   Answer. g is the function whose graph is the set $\{(y, x) \mid y \in (0, +\infty) \text{ and } x \in \mathbb{R} \text{ and } y = \exp(x)\}$.

   g is obtained when we 'flip' the picture for the coordinate plane diagram for function f below along a line parallel to the line $y = x$.