1. **Definition.**

Let A, B be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ be functions.

g is said to be an **inverse function** *of f if both of the following statements hold:*

(a) *For any $x \in A$, $(g \circ f)(x) = x$.* (b) *For any $y \in B$, $(f \circ g)(y) = y$.*

Definition.

Let C be a set.

Define the function $\mathsf{id}_C : C \longrightarrow C$ by $\mathsf{id}_C(z) = z$ for any $z \in C$.

id_C is called the **identity function on the set** *C.*

Remark 1 on the definition for the notion of inverse function.

By the respective definitions for the notions of inverse function, composition, and identity function:

$$g : B \longrightarrow A \text{ is an inverse function of } f : A \longrightarrow B$$

iff

($g \circ f = id_A$ and $f \circ g = id_B$ as functions).

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Remark 2 on the definition for the notion of inverse function.

See the 'symmetry' in the definition: simultaneously interchange

- 'A' and 'B',
- 'f' and 'g',
- 'x' and 'y'.

What do we get?

Consequence of this 'symmetry' in the definition:

$g : B \longrightarrow A$ is an inverse function of $f : A \longrightarrow B$

iff

$f : A \longrightarrow B$ is an inverse function of $g : B \longrightarrow A$.

Definition.

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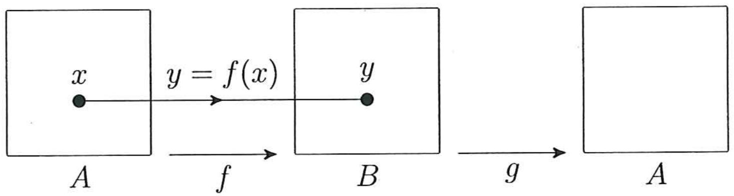
g is said to be an **inverse function** *of f if both of the following statements hold:*

(a) *For any $x \in A$, $(g \circ f)(x) = x$.* (b) *For any $y \in B$, $(f \circ g)(y) = y$.*

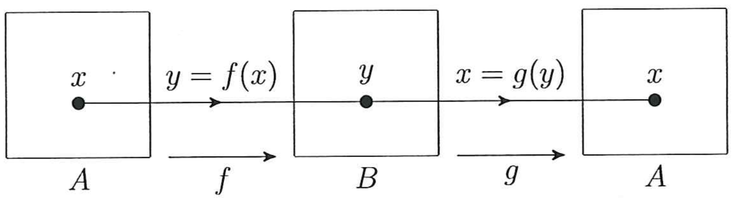
Remark 3 on the definition for the notion of inverse function.

How does such a function g 'interact' with f?

(a) Pick any $x \in A$. x is 'assigned' by f to $f(x)$.

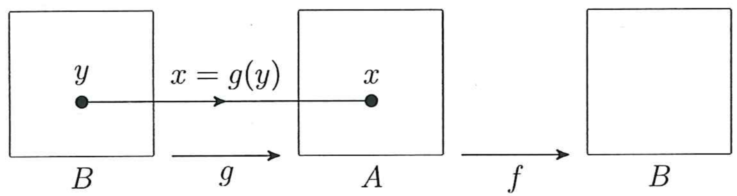


But then $f(x)$ is 'assigned' by g to x.

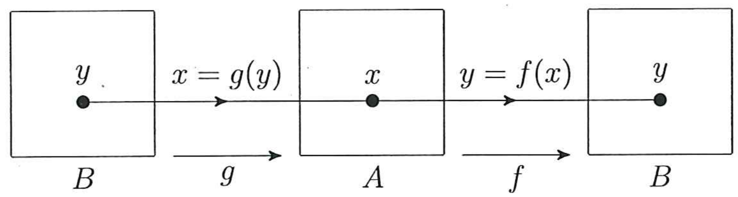


So g 'cancels' what f does to x.

(b) Pick any $y \in B$. y is 'assigned' by g to $g(y)$.



But then $g(y)$ is 'assigned' by f to y.



So f 'cancels' what g does to y.

Formal formulation: For any $x \in A$, for any $y \in B$, $(y = f(x)$ iff $x = g(y))$.

2. **Theorem (1). (Re-formulation of the definition of inverse function.)**

Let A, B be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ be functions.

The statements below are logically equivalent:

($\star_0$) *g is an inverse function of f.*

($\star_1$) *$g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$ as functions.*

($\star_2$) *f is an inverse function of g.*

($\star_3$) *For any $x \in A$, for any $y \in B$, ($y = f(x)$ iff $x = g(y)$).*

Proof of Theorem (1).

Let A, B be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ be functions.

Logical equivalence amongst ($\star_0$), ($\star_1$), ($\star_2$): see Remark 1, Remark 2 above.

- [($\star_0$) $\Longrightarrow$ ($\star_3$)?] Exercise: carefully formulate the idea in Remark 3 above.
- [($\star_3$) $\Longrightarrow$ ($\star_0$)?] Suppose that for any $x \in A$, $y \in B$,($y = f(x)$ iff $x = g(y)$).

  * Pick any $s \in A$. [Try to deduce: $(g \circ f)(s) = s$.]

    Define $u = f(s)$. We have $u \in B$. By assumption, $s = g(u)$. Then $(g \circ f)(s) = g(f(s)) = g(u) = s$.

  * Pick any $v \in B$. [Try to deduce: $(f \circ g)(v) = v$.]

    Define $t = g(v)$. We have $t \in A$. By assumption, $v = f(t)$. Then $(f \circ g)(v) = f(g(v)) = f(t) = v$.

  It follows that g is an inverse function of f.

3. **Theorem (2). (Uniqueness of inverse function.)**
Let A, B be sets, and $f : A \longrightarrow B$ be a function.
f has at most one inverse function.

Proof of Theorem (2).
Let A, B be sets, and $f : A \longrightarrow B$ be a function.
Suppose $g, h : B \longrightarrow A$ are inverse functions of f.
[We want to deduce that $g(y) = h(y)$ for any $y \in B$.]

Ask: What do we want to prove?
What is 'uniqueness'?
Answer. We want to prove:
- If there are two functions from B to A, both serving as inverse functions of f, then they are the same as each other as functions.

Pick any $y \in B$.
Define $x = g(y)$. We have $x \in A$.
Then $y = (f \circ g)(y)$
$= f(g(y)) = f(x)$.
Therefore $h(y) = h(f(x)) = (h \circ f)(x)$
$= x$
$= g(y)$.
It follows that $g = h$ as functions. □

4. **Definition.**

Let D, R be sets and $h : D \longrightarrow R$ be a function.

h is said to be **bijective** *if h is both surjective and injective.*

Remark.

Hence h is bijective iff both of the statements (S), (I) below hold:

(S): *For any $v \in R$, there exists some $u \in D$ such that $v = h(u)$.*

(I): *For any $u, t \in D$, if $h(u) = h(t)$ then $u = t$.*

5. **Theorem (3). (Necessary condition for existence of inverse function.)**

Let A, B be sets, $f : A \longrightarrow B$ be a function.

Suppose f has an inverse function, say, $g : B \longrightarrow A$.

Then each of f, g is bijective.

Proof of Theorem (3).

Let A, B be sets, $f : A \longrightarrow B$ be a function.

Suppose f has an inverse function, say, $g : B \longrightarrow A$.

- ['Is f surjective?'] [Want to deduce: For any $y \in B$, there exists some $x \in A$ such that $y = f(x)$.]

  Pick any $y \in B$. Define $x = g(y)$. By definition, $x \in A$.

  For the same $x \in A$, $y \in B$, we have $f(x) = f(g(y)) = (f \circ g)(y) = y$. Hence f is surjective.

- ['Is f injective?'] [Want to deduce: For any $x, w \in A$, if $f(x) = f(w)$ then $x = w$.]

  Pick any $x, w \in A$. Suppose $f(x) = f(w)$. Then

  $$x = (g \circ f)(x) = g(f(x)) = g(f(w)) = (g \circ f)(w) = w.$$

  Hence f is injective.

By definition, g is an inverse function of f. Then by Theorem (1), g has an inverse function, namely, f.

It follows from the argument above that g is both surjective and injective. □

Remark. The natural question to ask is: *Is the necessary condition sufficient?*

Answer. Yes, but...