1. The statement ($\sharp$) is true:

($\sharp$) *Let A, B be sets and $f : A \longrightarrow B$ be a function. For any subset U of B, $f(f^{-1}(U)) \subset U$.*

Proof of the statement ($\sharp$)?

- Let A, B be sets and $f : A \longrightarrow B$ be a function. Let U be a subset of B.

[Want to prove : $f(f^{-1}(U)) \subset U$.
This reads : 'For any object y, if $y \in f(f^{-1}(U))$ then $y \in U$.']

Pick any object y. Suppose $y \in f(f^{-1}(U))$. [Try to deduce : $y \in U$.]

By the definition of image sets,
there exists some $x \in f^{-1}(U)$ such that $y = f(x)$.

Now $x \in f^{-1}(U)$.

By the definition of pre-image sets,
there exists some $z \in U$ such that $z = f(x)$.

Then $y = f(x) = z \in U$.

It follows that $f(f^{-1}(U)) \subset U$. $\square$

2. The statement (b) is false:

(b) *Let A, B be sets and $f : A \longrightarrow B$ be a function. For any subset U of B, $f(f^{-1}(U)) \supset U$.*

Dis-proof of the statement (b)?

Negation of (b) reads:

'There exist some sets A, B, some function $f: A \to B$, some subset U of B such that $f(f^{-1}(U)) \not\supset U$.'

We give a counter-example for the dis-proof of (b):

- Take $A = \{0\}$, $B = \{1, 2\}$, $U = \{1, 2\}$.

  Note that $U \subset B$.

  Define the function $f: A \to B$ by $f(0) = 1$.

  Note that $f^{-1}(U) = \{0\}$ and $f(f^{-1}(U)) = \{1\}$.

  We have $2 \in U$ and $2 \notin f(f^{-1}(U))$.

  Then $U \not\subset f(f^{-1}(U))$. $\square$

3. Follow-up questions:

(a) Can we impose further assumption on f to make the conclusion in the statement (♭) hold?

(We are looking for some sufficient condition(s) for the statement (♭).)

It is a difficult question: how to conjure something out of nothing?

(b) What must happen to f if the conclusion in the statement (♭) is true?

(We are looking for some necessary condition(s) for the statement (♭).)

- Suppose that for any subset U of B, $f(f^{-1}(U)) \supset U$. [So what happens?]

[Can we name some sets which we know for sure are subsets of B?]

B is a subset of B. Then $f(f^{-1}(B)) \supset B$.

Note that $A = f^{-1}(B)$. [We need Theorem (1).]

Then $f(A) \supset B$.

Now, for any $y \in B$, we have $y \in f(A)$. For the same y, there exists some $x \in A$ such that $y = f(x)$. [We have used definition of image sets.]

It follows that f is surjective. $\square$

Follow-up questions:

(a) Can we impose further assumption on f to make the conclusion in the statement ($\flat$) hold?
(We are looking for some sufficient condition(s) for the statement ($\flat$).)

(b) What must happen to f if the conclusion in the statement ($\flat$) is true?
(We are looking for some necessary condition(s) for the statement ($\flat$).)

- Suppose that for any subset U of B, $f(f^{-1}(U)) \supset U$. Then f is surjective.

(c) Is the necessary condition sufficient?
We ask whether the statement below is true:

- Suppose f is surjective. Then for any subset U of B, $f(f^{-1}(U)) \supset U$.

Answer: Yes. Justification?

- Suppose f is surjective.
Pick any subset U of B. [Want to deduce: $U \subset f(f^{-1}(U))$.]
Pick any object y. Suppose $y \in U$. [Want to deduce: $y \in f(f^{-1}(U))$.]
We have $y \in B$. By surjectivity, there exists some $x \in A$ such that $y = f(x)$.
Since $y = f(x)$ and $y \in U$, we have $x \in f^{-1}(U)$. [We have used definition of pre-image sets.]
Since $y = f(x)$ and $x \in f^{-1}(U)$, we have $y \in f(f^{-1}(U))$. [We have used definition of image sets.]
It follows that $U \subset f(f^{-1}(U))$. $\square$

4. Conclusion in this investigation? The statement $(\star)$ holds:

$(\star)$ *Let A, B be sets and $f : A \longrightarrow B$ be a function. The following statements are equivalent:*

$(\star_1)$ *f is surjective.*

$(\star_2)$ *For any subset U of B, $f(f^{-1}(U)) \supset U$.*

$(\star_3)$ *For any subset U of B, $f(f^{-1}(U)) = U$.*

This is a characterization of the surjectivity of a function in terms of image sets and pre-image sets.

Question. What about a characterization of the injectivity of a function in terms of image sets and pre-image sets?

The statement $(\star')$ holds:

$(\star')$ *Let A, B be sets and $f : A \longrightarrow B$ be a function. The following statements are equivalent:*

$(\star'_1)$ *f is injective.*

$(\star'_2)$ *For any subset S of A, $f^{-1}(f(S)) \subset S$.*

$(\star'_3)$ *For any subset S of A, $f^{-1}(f(S)) = S$.*