1. Definitions.

Let A, B be sets and $f : A \longrightarrow B$ be a function from A to B.

(a) Let S be a subset of A. The **image set of the set S under the function f is** defined to be the set

$$\{y \in B : \text{There exists some } x \in S \text{ such that } y = f(x)\}.$$

It is denoted by $f(S)$.

(b) Let U be a subset of B. The **pre-image set of the set U under the function** f is defined to be the set

$$\{x \in A : \text{There exists some } y \in U \text{ such that } y = f(x)\}.$$

It is denoted by $f^{-1}(U)$.

2. Theorem (1).

Let A, B be sets, and $f : A \longrightarrow B$ be a function. The following statements hold:

(1a) $f(\emptyset) = \emptyset$.

(1b) $f^{-1}(\emptyset) = \emptyset$.

(1c) $f(A) \subset B$.

(1d) $f^{-1}(B) = A$.

(1b) Let $x \in A$. $f(\{x\}) = \{f(x)\}$.

(1c) Let $x \in A$, $y \in B$. The statements below are logically equivalent:

   (i) $x \in f^{-1}(\{y\})$.

   (ii) $f(x) \in \{y\}$.

   (iii) $f(x) = y$.

3. Theorem (2).

Let A, B be sets, and $f : A \longrightarrow B$ be a function. The following statements hold:

(2a) Let S, T be subsets of A. Suppose $S \subset T$. Then $f(S) \subset f(T)$.

(2b) Let H, K be subsets of A.

(1) $f(H \cup K) \supset f(H) \cup f(K)$.

(2) $f(H \cup K) \subset f(H) \cup f(K)$.

(3) $f(H \cup K) = f(H) \cup f(K)$.

(2c) Let H, K be subsets of A. $f(H \cap K) \subset f(H) \cap f(K)$.

4. Proof of Statement (2a) of Theorem (2).

Let A, B be sets and $f : A \longrightarrow B$ be a function.

Let S, T be subsets of A. Suppose $S \subset T$.

[We want to deduce that $f(S) \subset f(T)$.

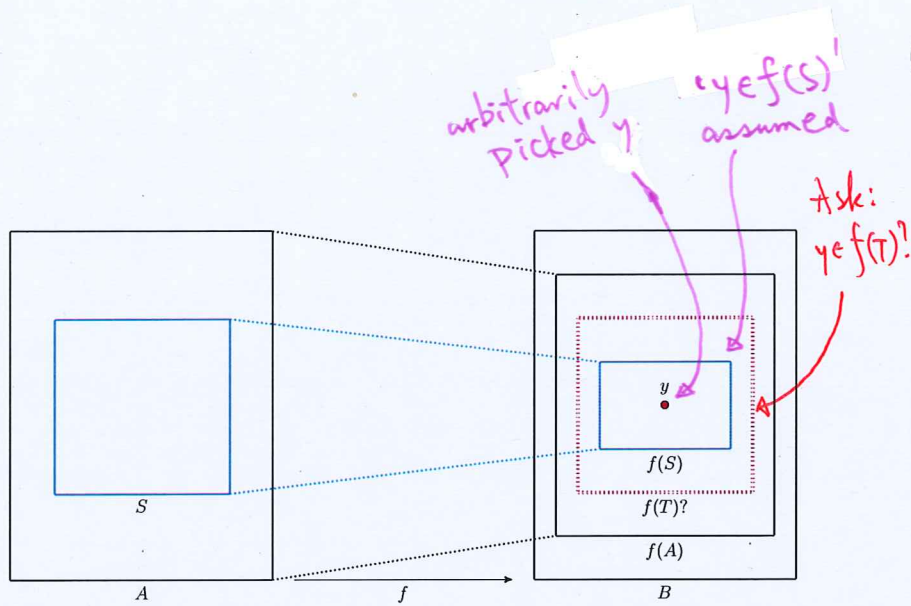What to do, really? We want to prove:

'For any object y, if $y \in f(S)$ then $y \in f(T)$.'
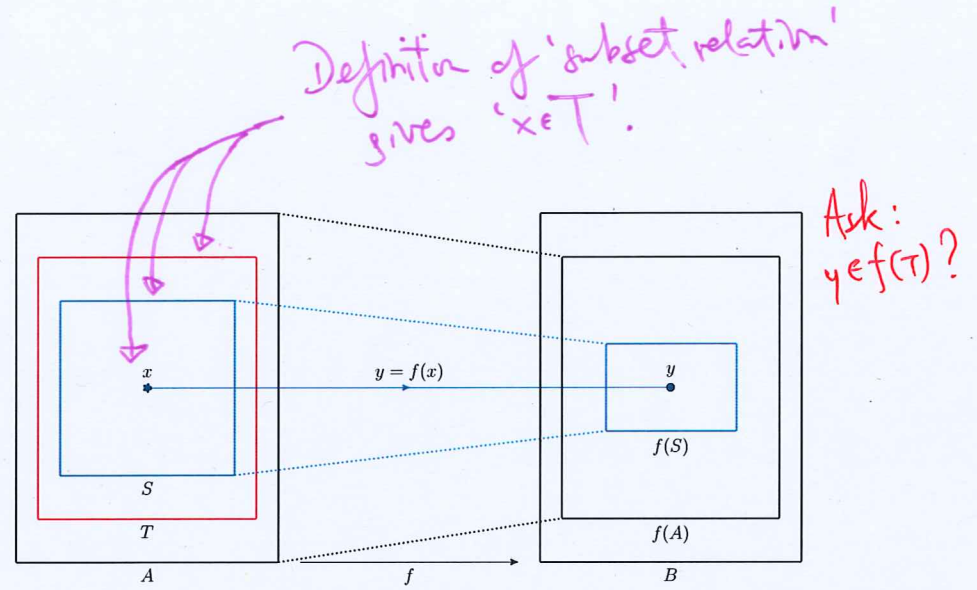
Think about this before proceeding any further.]

Proof of Statement (2a) of Theorem (2) .

... Suppose $S \subset T$. [Want to prove: $f(S) \subset f(T)$. This reads: 'for any y, if $y \in f(S)$ then $y \in f(T)$'.]
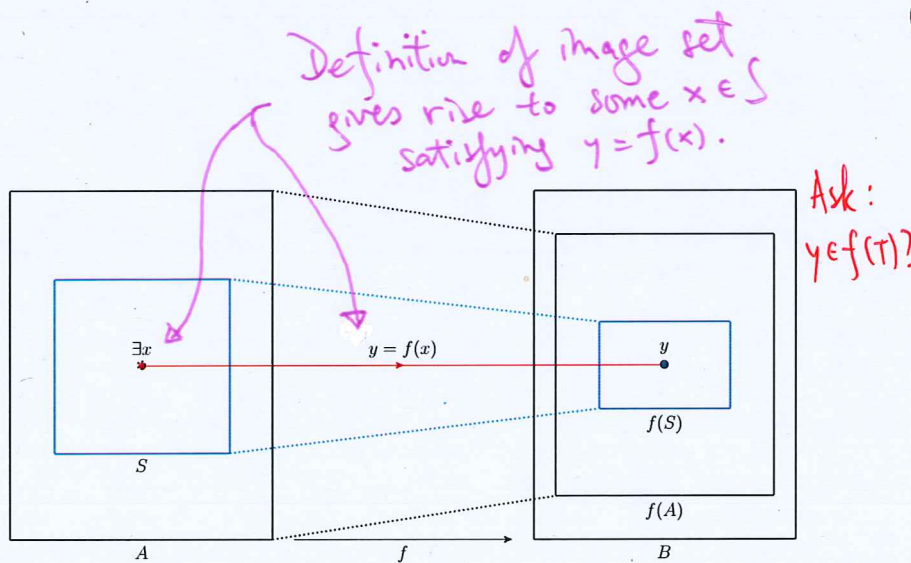
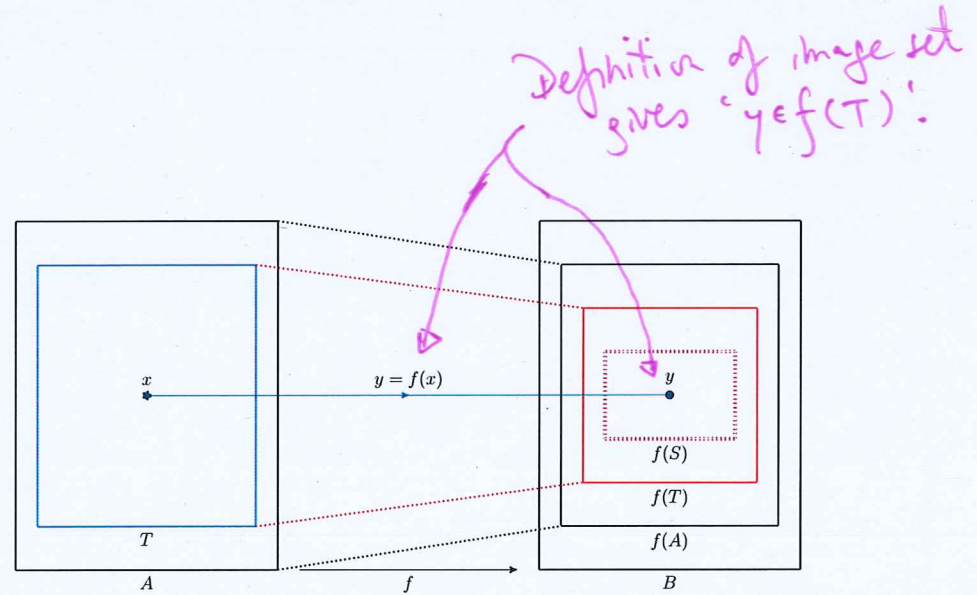**(1)**

arbitrarily picked y

'$y \in f(S)$' assumed

Ask: $y \in f(T)$?

$f(S)$

$f(T)$?

$f(A)$

S

A f B

(3)

Definition of 'subset relation' gives '$x \in T$'.

Ask: $y \in f(T)$?

x $y = f(x)$ y

S

T

$f(S)$

$f(A)$

A f B

(2)

Definition of image set gives rise to some $x \in S$ satisfying $y = f(x)$.

Ask: $y \in f(T)$?

$\exists x$ $y = f(x)$ y

$f(S)$

S

$f(A)$

A f B

(4)

Definition of image set gives '$y \in f(T)$'.

x $y = f(x)$ y

$f(S)$

T

$f(T)$

$f(A)$

A f B
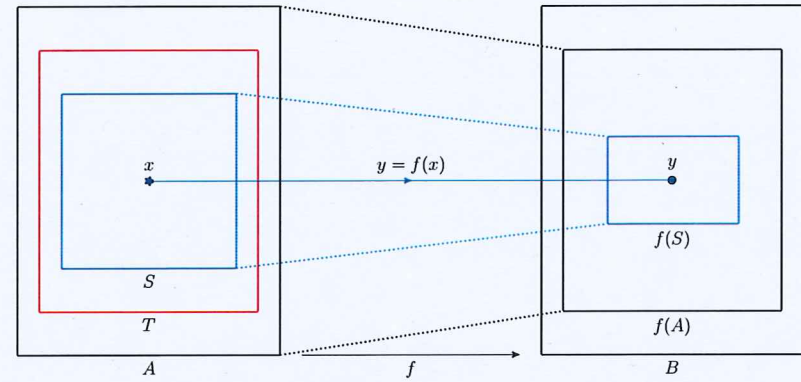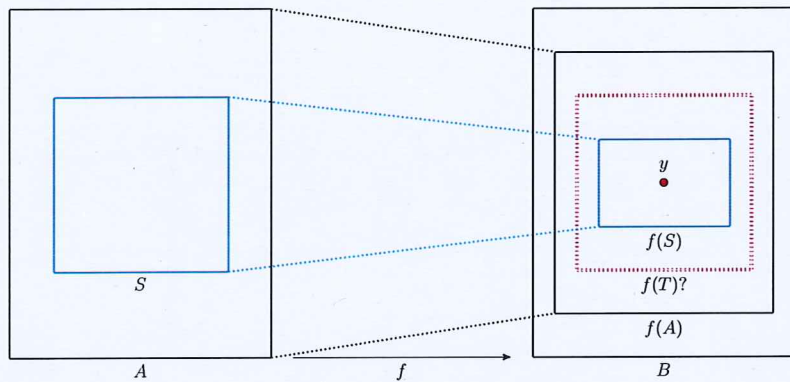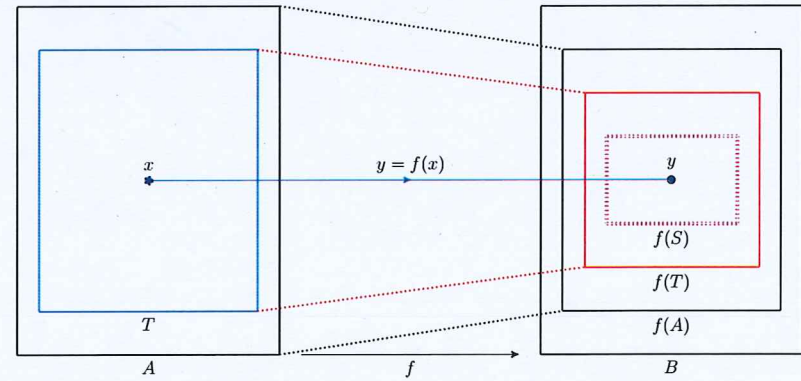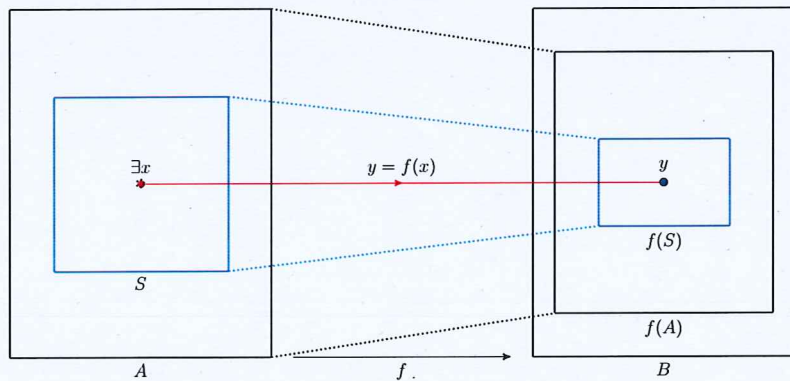
Proof of Statement (2a) of Theorem (2).

... Suppose $S \subset T$. [Want to prove: $f(S) \subset f(T)$. This reads: 'for any y, if $y \in f(S)$ then $y \in f(T)$'.]

(1) Pick any object y. [From now on, this y is fixed.]

Suppose $y \in f(S)$. [Want to deduce: $y \in f(T)$.]

(3) Since $x \in S$ and $S \subset T$, we have $x \in T$.

[Reason: Definition of notion of subset.]

(2) Then there exists some $x \in S$ [From now on, this x is fixed.] such that $y = f(x)$.

[Reason: Definition of image sets.]

(4) For the same x, y, since $x \in T$ and $y = f(x)$, we have $y \in f(T)$.

[Reason: Definition of image sets.]

It follows that $f(S) \subset f(T)$.

5. Proof of Statement (2b) of Theorem (2).

Let A, B be sets and $f : A \longrightarrow B$ be a function. Let H, K be subsets of A.

(1) [We want to prove that $f(H) \cup f(K) \subset f(H \cup K)$.]

Note that $H \subset H \cup K$. Then, by (2a), we have $f(H) \subset f(H \cup K)$.
Also note that $K \subset H \cup K$. Then, by (2a), we have $f(K) \subset f(H \cup K)$.
Since $f(H) \subset f(H \cup K)$ and $f(K) \subset f(H \cup K)$, we have $f(H) \cup f(K) \subset f(H \cup K)$.

(2) [We want to prove that $f(H \cup K) \subset f(H) \cup f(K)$.

What to do, really? We want to prove:

For any object y, if $y \in f(H \cup K)$ then $y \in f(H) \cup f(K)$.

Think about this before proceeding any further.]

Pick any object y. Suppose $y \in f(H \cup K)$.
Then, by the definition of image sets,
there exists some $x \in H \cup K$ such that $y = f(x)$.

Since $x \in H \cup K$, we have $x \in H$ or $x \in K$.
* (Case 1). Suppose $x \in H$.
Since $x \in H$ and $y = f(x)$, we have $y \in f(H)$.
Then $y \in f(H)$ or $y \in f(K)$. Therefore $y \in f(H) \cup f(K)$.

* (Case 2). Suppose $x \in K$.
[...]
Therefore $y \in f(H) \cup f(K)$.

Hence, in any case,
$y \in f(H) \cup f(K)$.
It follows that
$f(H \cup K) \subset f(H) \cup f(K)$. ▢

(3) By (2b1), (2b2), we have $f(H \cup K) = f(H) \cup f(K)$.

6. Proof of Statement (2c) of Theorem (2).

Let A, B be sets and $f : A \longrightarrow B$ be a function. Let H, K be subsets of A.

[We want to prove $f(H \cap K) \subset f(H) \cap f(K)$.]

- Note that $H \cap K \subset H$.

  Then, by (2a), we have $f(H \cap K) \subset f(H)$.

  Also note that $H \cap K \subset K$.

  Then, by (2a), we have $f(H \cap K) \subset f(K)$.

  Since $f(H \cap K) \subset f(H)$ and $f(H \cap K) \subset f(K)$,

  we have $f(H \cap K) \subset f(H) \cap f(K)$. $\quad \square$

7. Theorem (3).

Let A, B be sets, and $f : A \longrightarrow B$ be a function. The following statements hold:

(3a) Let U, V be subsets of B. Suppose $U \subset V$. Then $f^{-1}(U) \subset f^{-1}(V)$.

(3b) Let U, V be subsets of B.

    (1) $f^{-1}(U \cup V) \supset f^{-1}(U) \cup f^{-1}(V)$.

    (2) $f^{-1}(U \cup V) \subset f^{-1}(U) \cup f^{-1}(V)$.

    (3) $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$.

(3c) Let U, V subsets of B.

    (1) $f^{-1}(U \cap V) \subset f^{-1}(U) \cap f^{-1}(V)$.

    (2) $f^{-1}(U \cap V) \supset f^{-1}(U) \cap f^{-1}(V)$.

    (3) $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$.

8. Proof of Statement (3b2) of Theorem (3).

Let A, B be sets, and $f : A \longrightarrow B$ be a function. Let U, V be subsets of B.

[We want to prove that $f^{-1}(U \cup V) \subset f^{-1}(U) \cup f^{-1}(V)$.

What to do, really? We want to prove:

    'For any object x, if $x \in f^{-1}(U \cup V)$ then $x \in f^{-1}(U) \cup f^{-1}(V)$.'

Think about this before proceeding any further.]

Proof of Statement (3b2) of Theorem (3).

... Let U, V be subsets of B.

$\left[\begin{array}{l} \text{Want to prove}: \quad f^{-1}(U \cup V) \subset f^{-1}(U) \cup f^{-1}(V). \\ \quad \text{This reads}: \quad \text{'For any object } x, \text{ if } x \in f^{-1}(U \cup V) \text{ then } x \in f^{-1}(U) \cup f^{-1}(V).' \end{array} \right]$

Pick any object x.

Suppose $x \in f^{-1}(U \cup V)$.

Then, by the definition of pre-image sets, there exists some $y \in U \cup V$ such that $y = f(x)$.

Since $y \in U \cup V$, we have $y \in U$ or $y \in V$.

$*$ (Case 1). Suppose $y \in U$.

Since $y \in U$ and $y = f(x)$, we have $x \in f^{-1}(U)$.

Then $x \in f^{-1}(U)$ or $x \in f^{-1}(V)$.

Therefore $x \in f^{-1}(U) \cup f^{-1}(V)$.

$*$ (Case 2). Suppose $y \in V$. [...] Therefore $x \in f^{-1}(U) \cup f^{-1}(V)$.

Hence, in any case, we have $x \in f^{-1}(U) \cup f^{-1}(V)$.

It follows that $f^{-1}(U \cup V) \subset f^{-1}(U) \cup f^{-1}(V)$. $\square$

9. Remark.

Which of the statements is true? Which not?

(a) Let A, B be sets, and $f : A \longrightarrow B$ be a function.

Let S, T be subsets of A.

Suppose $f(S) \subset f(T)$. Then $S \subset T$.

(b) Let A, B be sets, and $f : A \longrightarrow B$ be a function.

Let U, V be subsets of B.

Suppose $f^{-1}(U) \subset f^{-1}(V)$. Then $U \subset V$.

(c) Let A, B be sets, and $f : A \longrightarrow B$ be a function.

Let H, K be subsets of A.

$f(H \cap K) \supset f(H) \cap f(K)$.

They are all false. (Can you provide counter-examples for the respective dis-proofs?)

10. **Theorem (4).**

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. The following statements hold:

(4a) Let S be a subset of A. $(g \circ f)(S) = g(f(S))$.

(4b) Let W be a subset of C. $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$.

11. **Proof of Statement (4b) of Theorem (4).**

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Let W be a subset of C.

[We want to prove the set equality $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$.

Hence we separate arguments into two parts, each on a 'subset relation'.

Which two?

(α) $(g \circ f)^{-1}(W) \subset f^{-1}(g^{-1}(W))$.

This reads:

'For any object x, if $x \in (g \circ f)^{-1}(W)$ then $x \in f^{-1}(g^{-1}(W))$.'

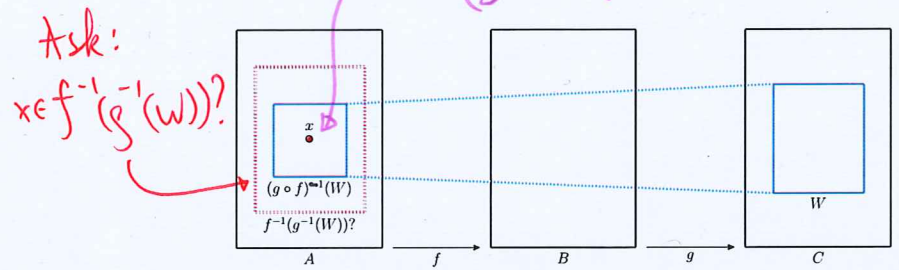(β) $(g \circ f)^{-1}(W) \supset f^{-1}(g^{-1}(W))$.

This reads:

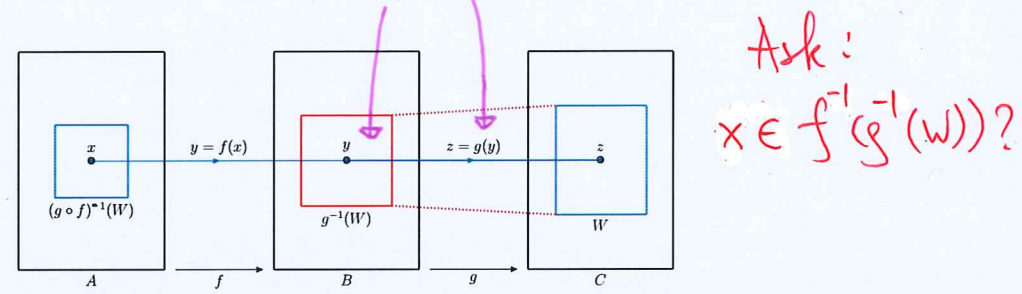'For any object x, if $x \in f^{-1}(g^{-1}(W))$ then $x \in (g \circ f)^{-1}(W)$.'

Think about this before proceeding any further.]

(α) [We are going to prove that $(g \circ f)^{-1}(W) \subset f^{-1}(g^{-1}(W))$.]

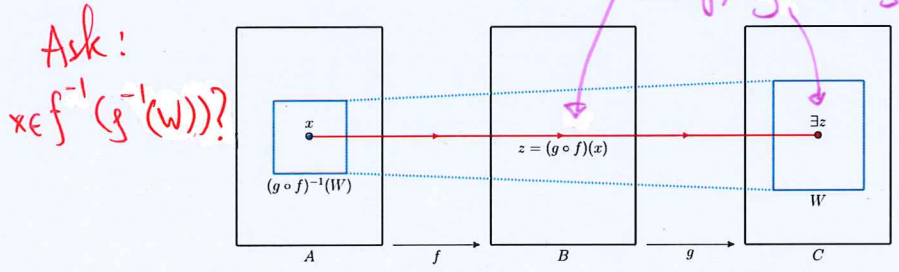For any x, if $x \in (g \circ f)^{-1}(W)$ then $x \in f^{-1}(g^{-1}(W))$.

(1)

arbitrarily picked x.
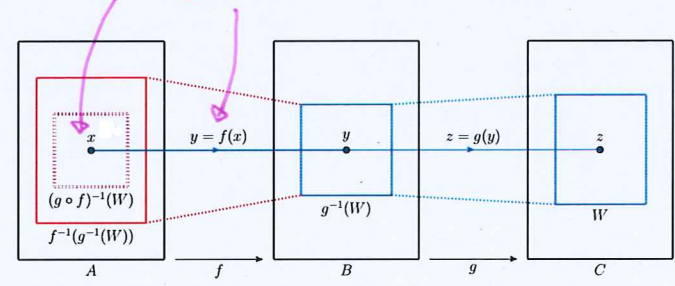'$x \in (g \circ f)^{-1}(W)$' assumed.

Ask:
$x \in f^{-1}(g^{-1}(W))$?
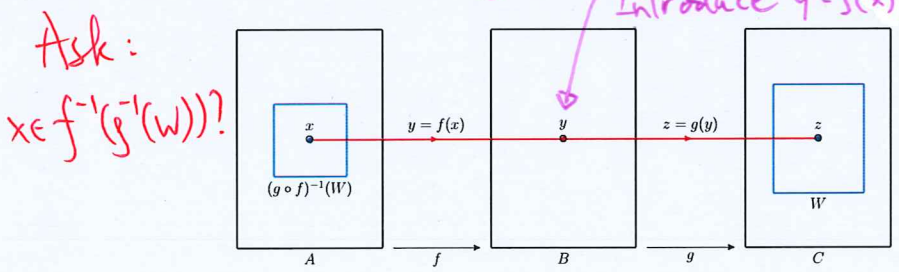

(2)

Definition of pre-image sets
gives rise to some $z \in W$
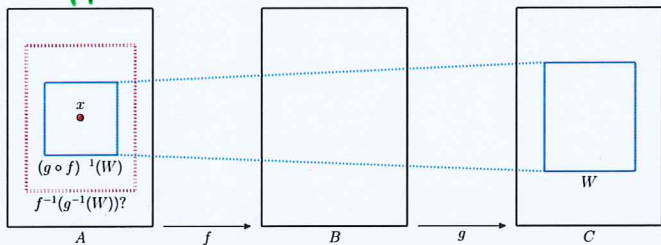satisfying $z = (g \circ f)(x)$.

Ask:
$x \in f^{-1}(g^{-1}(W))$?

(3)

Definition of composition
gives $z = g(f(x))$.
Introduce $y = f(x)$.

Ask:
$x \in f^{-1}(g^{-1}(W))$?

(4)

Definition of pre-image sets
gives $y \in g^{-1}(W)$.

Ask:
$x \in f^{-1}(g^{-1}(W))$?

(5)

Definition of pre-image sets
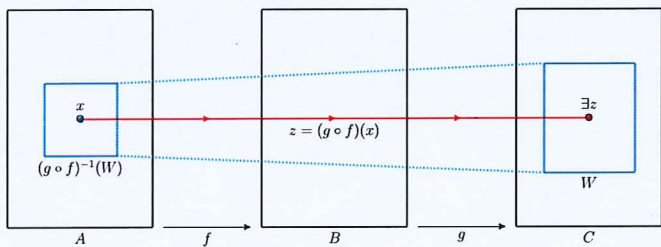gives $x \in f^{-1}(g^{-1}(W))$.

(α) [We are going to prove that $(g \circ f)^{-1}(W) \subset f^{-1}(g^{-1}(W))$.]

[For any x, if $x \in (g \circ f)^{-1}(W)$ then $x \in f^{-1}(g^{-1}(W))$.]
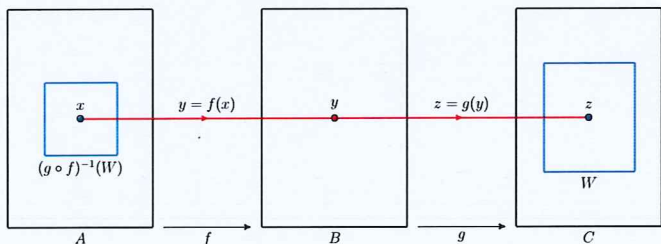
(1) Pick any object x.
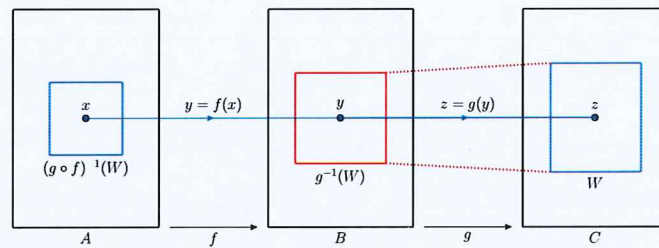Suppose $x \in (g \circ f)^{-1}(W)$.

(2) Then there exists some $z \in W$
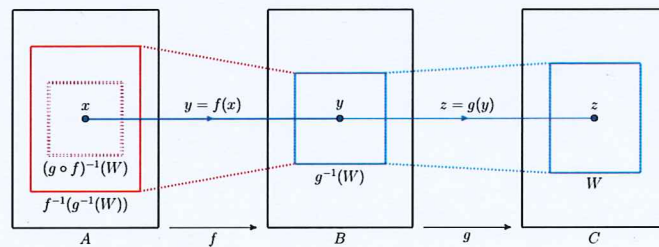such that $z = (g \circ f)(x)$.

(3) We have $z = g(f(x))$.
Take $y = f(x)$. Note that $y \in B$.

(4) We have $g(y) = g(f(x)) = z$ and $z \in W$.
Then $y \in g^{-1}(W)$.

(5) We have $f(x) = y$ and $y \in g^{-1}(W)$.
Then $x \in f^{-1}(g^{-1}(W))$.

It follows that
$$(g \circ f)^{-1}(W) \subset f^{-1}(g^{-1}(W)).$$

Proof of Statement (4b) of Theorem (4).

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Let W be a subset of C.

[We want to prove $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$.]

(α) ... It follows that $(g \circ f)^{-1}(W) \subset f^{-1}(g^{-1}(W))$.

(β) [We are going to prove that $(g \circ f)^{-1}(W) \supset f^{-1}(g^{-1}(W))$.]

$\left[\begin{array}{l} \text{For any } x, \text{ if } x \in f^{-1}(g^{-1}(W)) \\ \quad \text{then } x \in (g \circ f)^{-1}(W). \end{array} \right]$

Pick any object x.

Suppose $x \in f^{-1}(g^{-1}(W))$.

Then, by the definition of pre-image sets, there exists some $y \in g^{-1}(W)$ such that $y = f(x)$.

Now $y \in g^{-1}(W)$.

Then, by the definition of pre-image sets, there exist some $z \in W$ such that $z = g(y)$.

We have $z \in W$ and $z = g(y) = g(f(x)) = (g \circ f)(x)$. Then $x \in (g \circ f)^{-1}(W)$.

It follows that $f^{-1}(g^{-1}(W)) \subset (g \circ f)^{-1}(W)$.

It follows that $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$.

12. **Theorem (5).**

Let A, B be sets, and $f : A \longrightarrow B$ be a function. The following statements hold:

(5a) Let S be a subset of A. $f^{-1}(f(S)) \supset S$.

(5b) Let U be a subset of B. $f(f^{-1}(U)) \subset U$.

(5c) Let S be a subset of A. $f(f^{-1}(f(S))) = f(S)$.

(5d) Let U be a subset of B. $f^{-1}(f(f^{-1}(U))) = f^{-1}(U)$.

(5e) Let S be a subset of A, and U be a subset of B. $f(S \cap f^{-1}(U)) = f(S) \cap U$.

13. Definition.

Let A, B be sets and $f : A \longrightarrow B$ be a function. f is said to be **surjective** if the statement (S) holds:

(S): For any $y \in B$, there exists some $x \in A$ such that $y = f(x)$.

Theorem (6). (Characterizations of surjectivity).

Let A, B be sets and $f : A \longrightarrow B$ be a function.

The following statements are equivalent:

(I) f is surjective.   (Ia) $f(A) = B$.   (Ib) $f(A) \supset B$.

(II) For any subset U of B, $f(f^{-1}(U)) \supset U$.

(IIa) For any subset U of B, $f(f^{-1}(U)) = U$.

(III) For any subset U of B, there exists some subset S of A such that $U = f(S)$.

(IV) For any subset T of A, $f(A \backslash T) \supset B \backslash f(T)$.

(V) For any subset U, V of B, if $f^{-1}(U) \subset f^{-1}(V)$ then $U \subset V$.

(VI) For any subset U, V of B, if $f^{-1}(U) = f^{-1}(V)$ then $U = V$.

Proof of Theorem (6)?

14. Definition.

Let A, B be sets and $f : A \longrightarrow B$ be a function. f is said to be **injective** if the statement (I) holds:

(I): For any $x, w \in A$, if $f(x) = f(w)$ then $x = w$.

Theorem (7). (Characterizations of injectivity).

Let A, B be sets and $f : A \longrightarrow B$ be a function.

The following statements are equivalent:

(I) f is injective.

(II) For any subset S of A, $f^{-1}(f(S)) \subset S$.

(IIa) For any subset S of A, $f^{-1}(f(S)) = S$.

(III) For any subset S of A, there exists some subset U of B such that $S = f^{-1}(U)$.

(IV) For any subset S, T of A, $f(S \cap T) \supset f(S) \cap f(T)$.

(IVa) For any subset S, T of A, $f(S \cap T) = f(S) \cap f(T)$.

(V) For any subsets S, T of A, if $f(S) \subset f(T)$ then $S \subset T$.

(VI) For any subsets S, T of A, if $f(S) = f(T)$ then $S = T$.

15. Theorem (8).

Let A, B be sets and $f : A \longrightarrow B$ be a function.

(8a) Let $\{U_n\}_{n=0}^{\infty}$ be an infinite sequence of subsets of B. $(\{f^{-1}(U_n)\}_{n=0}^{\infty}$ is an infinite sequence of subsets of A.) The following statements hold:

(1) $\quad f^{-1}\left(\bigcap_{n=0}^{\infty} U_n\right) = \bigcap_{n=0}^{\infty} f^{-1}(U_n)$.

(2) $\quad f^{-1}\left(\bigcup_{n=0}^{\infty} U_n\right) = \bigcup_{n=0}^{\infty} f^{-1}(U_n)$.

(8b) Let $\{S_n\}_{n=0}^{\infty}$ be an infinite sequence of subsets of A. $(\{f(S_n)\}_{n=0}^{\infty}$ is an infinite sequence of subsets of B.) The following statements hold:

(1) $\quad f\left(\bigcap_{n=0}^{\infty} S_n\right) \subset \bigcap_{n=0}^{\infty} f(S_n)$.

(2) $\quad f\left(\bigcup_{n=0}^{\infty} S_n\right) = \bigcup_{n=0}^{\infty} f(S_n)$.

(8c) The statements below are logically equivalent:

(i) f is injective.

(ii) For any infinite sequence of subsets $\{S_n\}_{n=0}^{\infty}$ of A, $f\left(\bigcap_{n=0}^{\infty} S_n\right) = \bigcap_{n=0}^{\infty} f(S_n)$.