1. **Definition.**

   Suppose $f : D \longrightarrow \mathbb{R}$ is a function, whose domain D is a subset of $\mathbb{R}^n$.

   For each point $c \in \mathbb{R}$, the set $f^{-1}(\{c\})$ is called the **level set** *of f at c.*

   Remark. By definition, $f^{-1}(\{c\}) = \{x \in \mathbb{R}^n : f(x) = c\}$. Hence the level set of f at c is the solution set of the equation $f(u) = c$ with unknown u in $\mathbb{R}^n$,

2. **Curves as level sets.**

   Suppose $n = 2$. Suppose D is a 'nice' subset of $\mathbb{R}^2$ (for example, an open subset of $\mathbb{R}^2$), and f is 'nice' (for example, being continuously differentiable, and with 'very few' 'zeros' in its gradient).

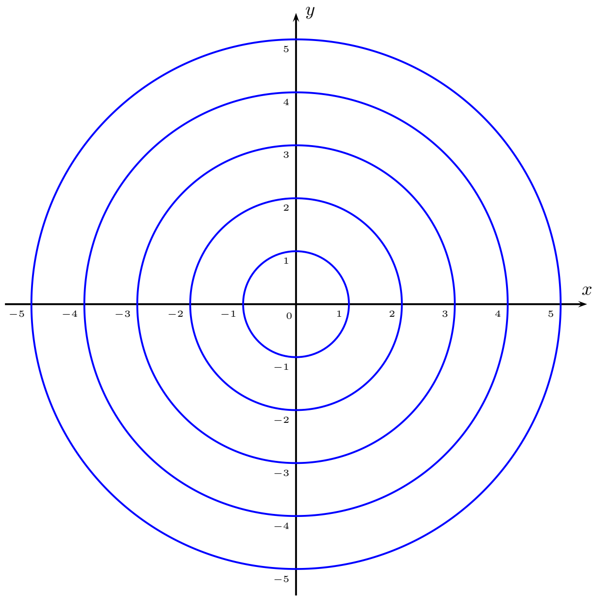
   Because f is so 'nice', 'many' a non-empty level set $f^{-1}(\{c\})$ will also look 'nice' (for example, appearing as a 'nice' 'continuous curve') on $\mathbb{R}^2$. We can draw the various level sets of such a function f on $\mathbb{R}^2$. Such a picture resembles a 'contour map' in an atlas which displays the shape of the landscape of a region by showing the contours of equal altitude. Through such a picture we can visualize the graph of f.

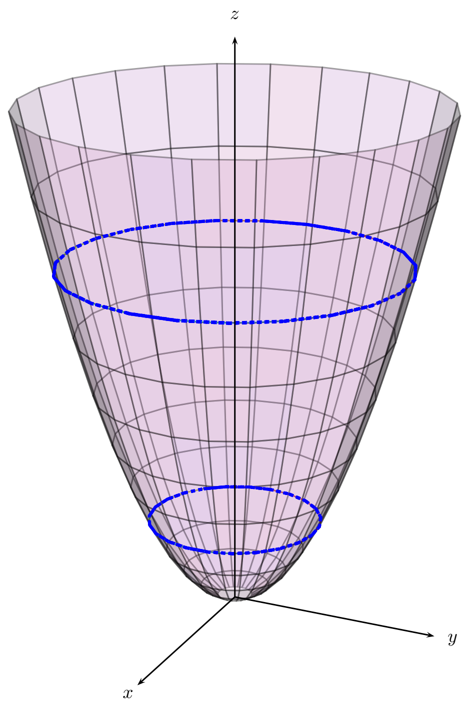
3. **Examples of curves as level sets.**

   (a) *Define the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $f(x, y) = x^2 + y^2$ for any $x, y \in \mathbb{R}$.*

   What are its level sets? How does the 'contour map' look like? How does the graph of f look like?

   - When $c > 0$, $f^{-1}(\{c\})$ is the circle with radius $\sqrt{c}$ centred at origin.
   - When $c < 0$, $f^{-1}(\{c\}) = \emptyset$.
   - $f^{-1}(\{0\}) = \{(0, 0)\}$.





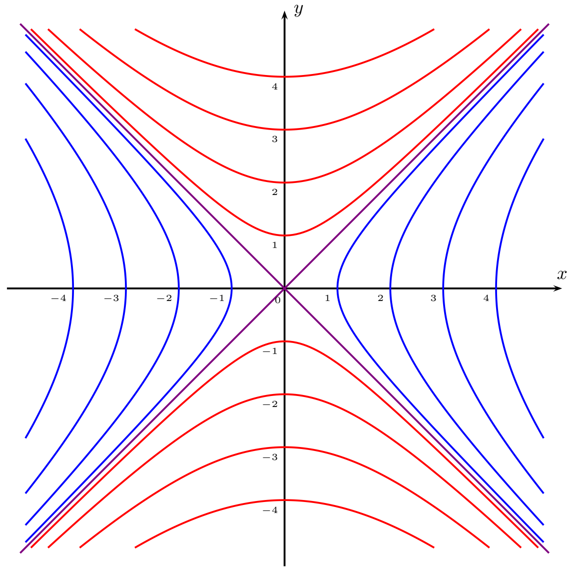
   The 'contour map' shows the family of concentric circles $\gamma_c : x^2 + y^2 = c$ with common centre $(0, 0)$, including the 'degenerate case' $x^2 + y^2 = 0$.

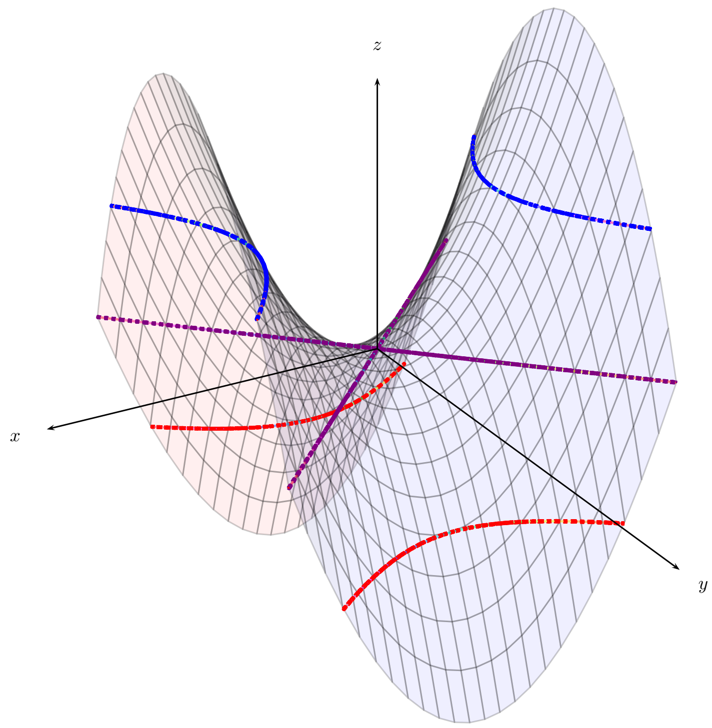
   The graph of f is the circular paraboloid $z = x^2 + y^2$ with the z-axis being the axis of symmetry.

(b) Define the function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $g(x, y) = x^2 - y^2$ for any $x, y \in \mathbb{R}$.

What are its level sets? How does the 'contour map' look like? How does the graph of g look like?

- When $c > 0$, $g^{-1}(\{c\})$ is the hyperbola $x^2 - y^2 = c$.
- When $c < 0$, $g^{-1}(\{c\})$ is the hyperbola $-x^2 + y^2 = -c$.
- $g^{-1}(\{0\})$ is the pair of straight lines $y = x$, $y = -x$.





The 'contour map' shows the family of hyperbolae $\eta_c : x^2 - y^2 = c$ with common centre $(0, 0)$, including the 'degenerate case' $x^2 - y^2 = 0$.

The graph of g is the hyperbolic paraboloid $z = x^2 - y^2$, which looks like a 'saddle', 'going up' on both sides of the x-axis and 'going down' on both sides of the y-axis.

4. **Surfaces as level sets.**

Suppose $n = 3$. Suppose D is a 'nice' subset of $\mathbb{R}^3$ (for example, an open subset of $\mathbb{R}^3$), and f is 'nice' (for example, being continuously differentiable, and with a Jacobian matrix which is full-rank throughout D except at a few points of D).

Because f is so 'nice', 'many' a non-empty level set $f^{-1}(\{c\})$ will also look 'nice' (for example, appearing as a 'nice' surface) in $\mathbb{R}^3$. We can draw the various level sets of such a function f on $\mathbb{R}^3$. Through such a picture we can visualize the graph of f.

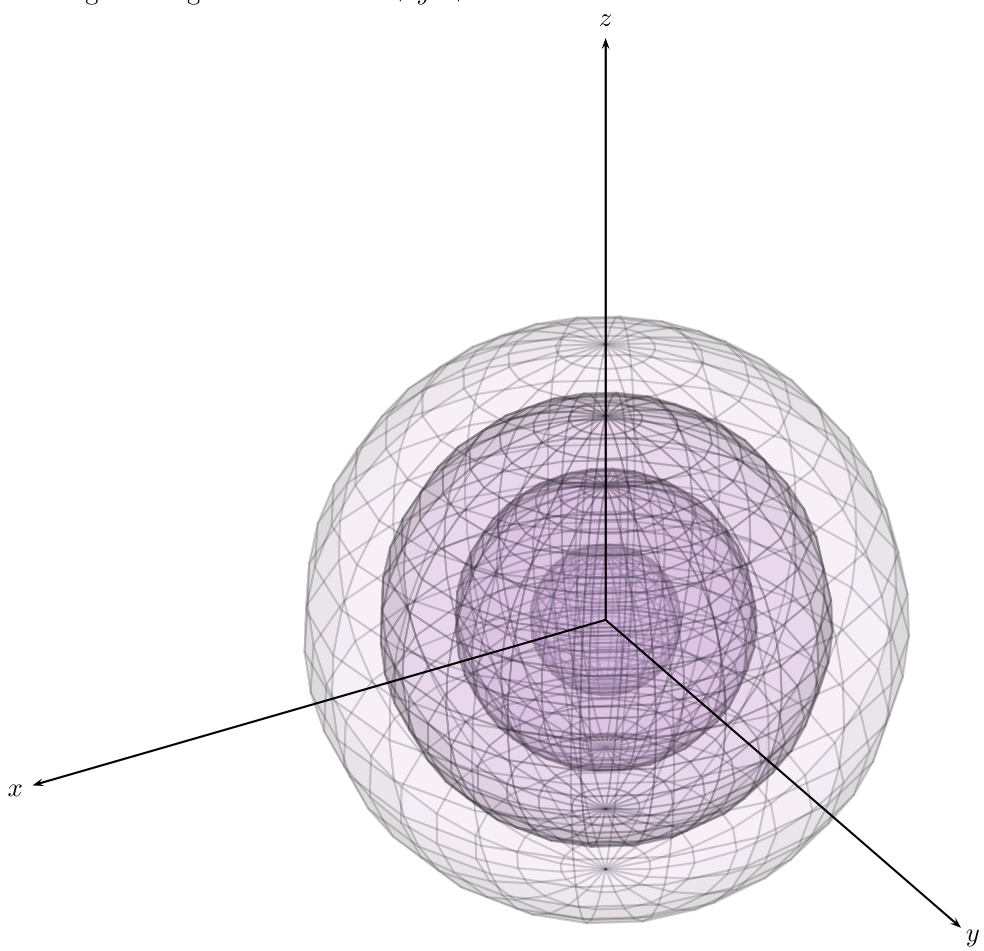
5. **Examples of surfaces as level sets.**

(a) *Define the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ by $f(x, y, z) = x^2 + y^2 + z^2$ for any $x, y, z \in \mathbb{R}$.*

What are the level sets of f? How does the 'contour map' look like?

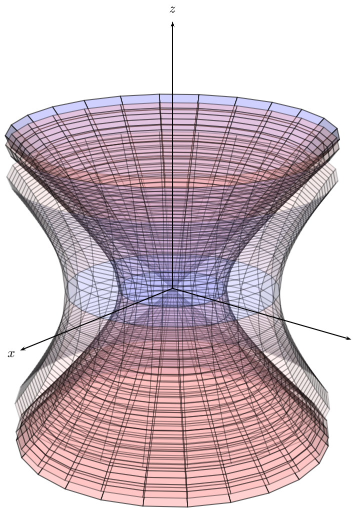
- When $c > 0$, $f^{-1}(\{c\})$ is the sphere with radius $\sqrt{c}$ centred at origin.
- When $c < 0$, $f^{-1}(\{c\}) = \emptyset$.
- $f^{-1}(\{0\}) = \{(0, 0, 0)\}$.

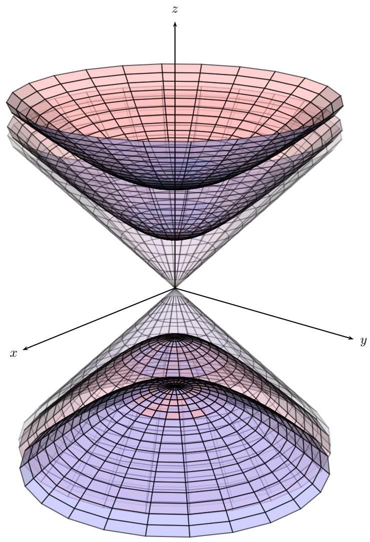
The 'contour map' shows the family of concentric spheres $\sigma_c : x^2 + y^2 + z^2 = c$ with common centre $(0, 0, 0)$, including the 'degenerate case' $x^2 + y^2 + z^2 = 0$.



(b) *Define the function* $g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ *by* $g(x, y, z) = x^2 + y^2 - z^2$ *for any* $x, y, z \in \mathbb{R}$.
What are the level sets of g? How does the 'contour map' look like?

- $g^{-1}(\{0\})$ is a cone with apex at origin, obtained by rotating about the z-axis the line $z = x$ on the xz-plane.
- When $c > 0$, $g^{-1}(\{c\})$ is the hyperboloid of one sheet, obtained by rotating about the z-axis the hyperbola $x^2 - z^2 = c$ on the xz-plane.
- When $c < 0$, $g^{-1}(\{c\})$ is a hyperboloid of two sheets, obtained by rotating about the z-axis the hyperbola $-x^2 + z^2 = -c$ on the xz-plane.





6. **Appendix: quadrics.**

Let Q *be an* $m \times m$*-symmetric matrix,* P *be an* $m \times 1$*-matrix,* R *be a real number, and* $h : \mathbb{R}^m \longrightarrow \mathbb{R}$ *be the function defined by* $h(x) = x^t Q x + P^t x + R$ *for any* $x \in \mathbb{R}^m$*. The pre-image set* $h^{-1}(\{0\})$ *is called a* **quadric**.

Examples of quadrics.

- $m = 2$.
  Ellipses (including circles), parabolae, hyperbolae; pairs of straight lines.
  These are 'curves': they are 'one-dimensional' geometric objects 'sitting' in a 'two-dimensional space'.
- $m = 3$.
  Ellipsoids (including spheres), paraboloids, hyperboloids; cylinders, cones.
  These are 'surfaces': they are 'two-dimensional' geometric objects 'sitting' in a 'three-dimensional space'.
- The set of all 'infinite' straight lines in the 'infinite' space can be viewed as the points on a quadric known as the Klein quadric.
  It turns out to be a 'four-dimensional' geometric object 'sitting' in a 'five dimensional space'.

Differential geometry and *algebraic geometry* begin with the study of these geometric objects, using tools from calculus and algebra respectively.