MATH1050 Parametrizations for curves and surfaces

1. In school maths we encountered the notion of locus for a 'point particle' 'moving' in a plane or in the space. For instance, consider the passage below:

   - *A 'point particle' is moving in the plane from time a to time b. At any time t, its position in the plane is given by the coordinates $(u(t), v(t))$.* $\cdots$

   Translated into the the language of function, what this passage is about is the function $f : [a, b] \longrightarrow \mathbb{R}^2$ given $f(t) = (u(t), v(t))$ for any $t \in [a, b]$. The position of the 'point particle' 'at time t' is the value of $f(t)$. The locus of the 'point particle' is the image set of $[a, b]$ under f.

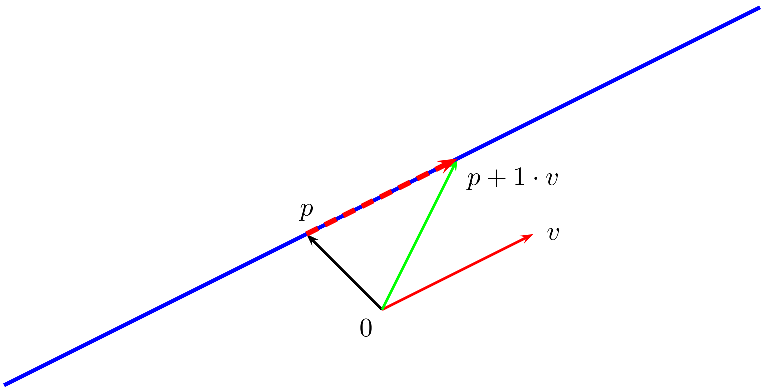
   So we did actually come across the notion of image set under a function, without knowing its name.

2. **Parametrized curves**.
   Suppose we are given a 'nice' curve C in $\mathbb{R}^n$. To 'describe' it we may think of the curve C (except perhaps a few points) as the image set of some subset I of $\mathbb{R}$, under some function f of one real variable. Very often I is so nice that it is an interval. f is so nice that it is differentiable, (or even better, smooth), and when it is restricted to I, it is injective.

   In this sense we say we are 'parametrizing' the curve C; the 'parametrization' under question is the function f. Equivalently C is the 'trace' of the 'parametrized curve' f. When the points in the set I is interpreted as time, the curve C is the locus of some 'point particle' moving in the plane/space according to some rule specified by the function f.

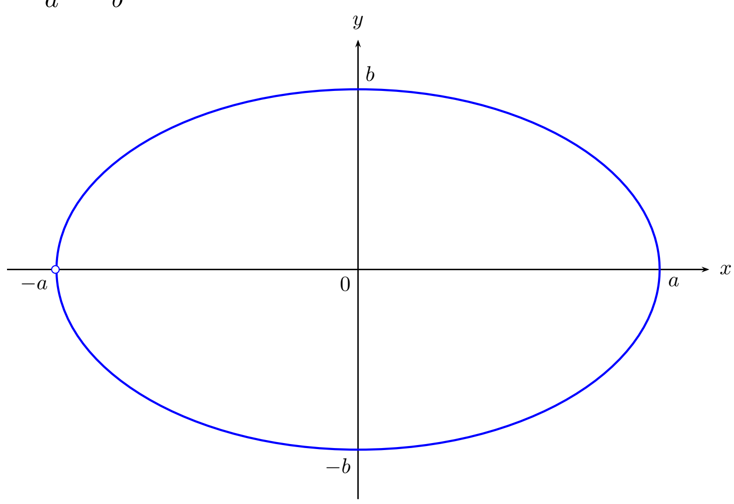
3. **Examples of parametrized curves**.

   (a) Let $p \in \mathbb{R}^n$, $v \in \mathbb{R}^n \backslash \{0\}$. (Think of p, v as vectors.)
   Define the function $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ by $f(t) = p + tv$ for any $t \in \mathbb{R}$.
   $f(\mathbb{R})$ is the **'infinite' straight line** in $\mathbb{R}^n$ passing through p and being parallel to v.



   Suppose $a, b \in \mathbb{R}$, and $a < b$. $f([a, b])$ is the line segment joining the points $p + av$, $p + bv$.

   (b) Let $a, b \in (0, +\infty)$.
   Define the function $g : (-\pi, \pi) \longrightarrow \mathbb{R}^2$ by $g(t) = (a\cos(t), b\sin(t))$ for any $t \in (-\pi, \pi)$.
   g 'parametrizes' the **ellipse** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with the point $(-a, 0)$ removed.

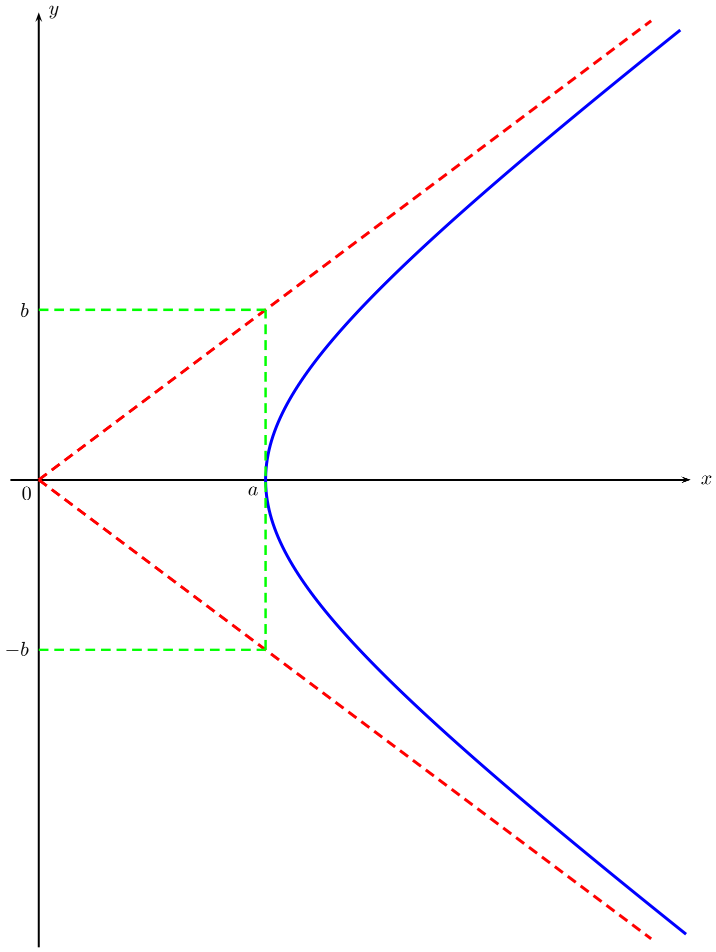


   Suppose $\sigma, \tau \in (-\pi, \pi)$, and $\sigma < \tau$. $g([\sigma, \tau])$ is the 'elliptic arc' of 'between $t = \sigma$ and $t = \tau$'.

(c) Let $a, b \in (0, +\infty)$.

Define the function $g : \mathbb{R} \longrightarrow \mathbb{R}^2$ by $g(t) = (a\cosh(t), b\sinh(t))$ for any $t \in \mathbb{R}$.

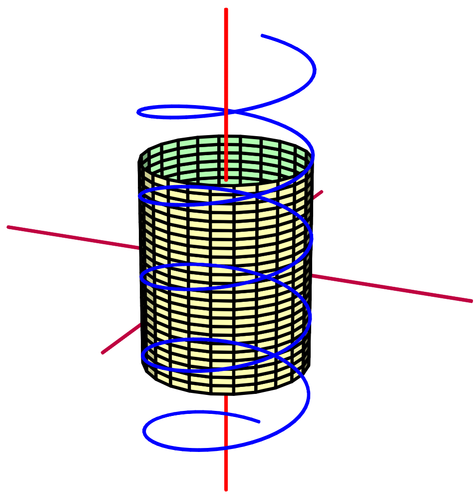
g 'parametrizes' one branch of the **hyperbola** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.



Suppose $\sigma, \tau \in (-\pi, \pi]$, and $\sigma < \tau$. $g([\sigma, \tau])$ is the 'hyperbolic arc' of 'between $t = \sigma$ and $t = \tau$'.

(d) Define the function $h : \mathbb{R} \longrightarrow \mathbb{R}^3$ by $h(t) = (\cos(2\pi t), \sin(2\pi t), t)$ for any $t \in \mathbb{R}$.

h 'parametrizes' the **circular helix** passing through the point $(1, 0, 0)$ with gauge 1 on the cylinder $x^2 + y^2 = 1$.



Suppose $a, b \in \mathbb{R}$ and $a < b$. $g([a, b])$ is the 'section' of this space curve 'between $t = a$ and $t = b$'.

4. **Parametrized surfaces**.

The idea in the notion of parametrized curves can be generalized to give the notion of parametrized surfaces (and beyond).

Suppose we are given a 'nice' surface S in $\mathbb{R}^n$. To 'describe' it we may think of the surface S (except perhaps a few points or a few curves on it) as the image set of some subset D of $\mathbb{R}^2$ under some function f of two real variables.

Very often D is so nice that it is a plane figure that we know well, for instance, a rectangular region, or a triangular region, or a disc. f is so nice that it is differentiable everywhere (or even better, smooth), and when it is restricted to D, it is injective.

In this sense we say we are 'parametrizing' the surface S; the 'parametrization' under question is the function f. Equivalently S is the 'trace' of the 'parametrized surface' f.

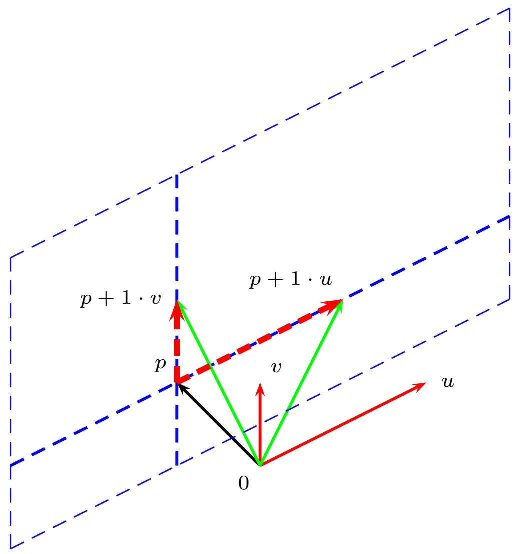
The restriction of f to any specific constant value, say, x_0, of its 'first real variable', defines the parametrized curve whose 'formula of definition' is given by $y \longmapsto f(x_0, y)$ whenever $(x_0, y) \in D$.

The restriction of f to any specific constant value, say, y_0, of its 'second real variable', defines the parametrized curve whose 'formula of definition' is given by $x \longmapsto f(x, y_0)$ whenever $(x, y_0) \in D$.

In this way we can visualize the surface S as the unions of two 'families of parametrized curves', obtained from restricting f to various specific values of its 'first real variable' and its 'second real variable' respectively.

5. **Examples of parametrized surfaces**.

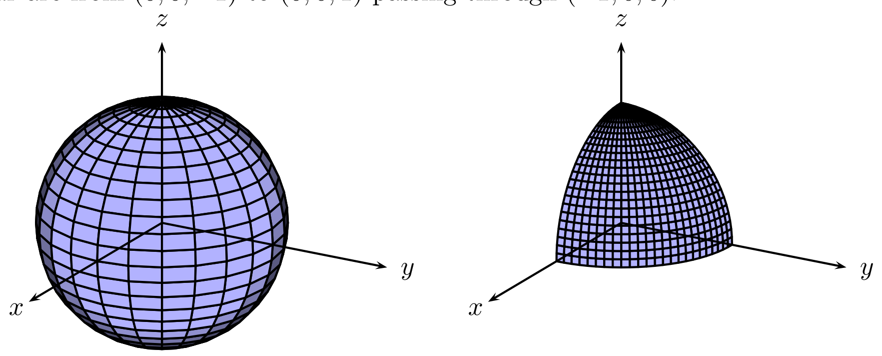
   (a) Let $p \in \mathbb{R}^n$, $u, v \in \mathbb{R}^n \backslash \{0\}$. Suppose u, v are linearly independent.
   Define the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ by $f(s,t) = p + su + tv$ for any $s, t \in \mathbb{R}$.
   $f(\mathbb{R}^2)$ is the **'infinite' plane** in $\mathbb{R}^n$ passing through p and being parallel to 'span' of u, v. The vector $u \times v$ is perpendicular to this plane.



   (b) Define the function $g : (-\pi, \pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}^3$ by

$$g(\theta, \varphi) = (\cos(\theta)\cos(\varphi), \sin(\theta)\cos(\varphi), \sin(\varphi))$$

   for any $\theta \in (-\pi, \pi)$, $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
   g 'parametrizes' the 'slitted' unit **sphere** $x^2 + y^2 + z^2 = 1$ with centre at origin.
   The 'slit' is the circular arc from $(0, 0, -1)$ to $(0, 0, 1)$ passing through $(-1, 0, 0)$.



   $g([0, \pi/2] \times [0, \pi/2))$ is one quarter of the 'northern' hemisphere.

(c) Let $r > 0$ and $R > 0$. Suppose $R > r$.

Define the function $h : (-\pi, \pi) \times (-\pi, \pi) \longrightarrow \mathbb{R}^3$ by

$$h(\theta, \varphi) = ((R + r\cos(\varphi))\cos(\theta), (R + r\cos(\varphi))\sin(\theta), r\sin(\varphi))$$

for any $\theta \in (-\pi, \pi)$, $\varphi \in (-\pi, \pi)$.

h 'parametrizes' the 'slitted' **torus** $(\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2$ with centre at origin, 'floated' on the xy-plane, obtained by rotating about the z-axis the circle on the xz-plane with centre at $(R, 0, 0)$ and with radius r.

The 'slits' are the union of two circles: the circle on the xz-plane with centre at $(-R, 0, 0)$ and with radius r, and the circle on the xy-plane with centre at $(0, 0, 0)$ and with radius $R - r$.