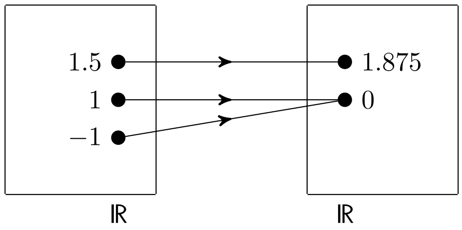
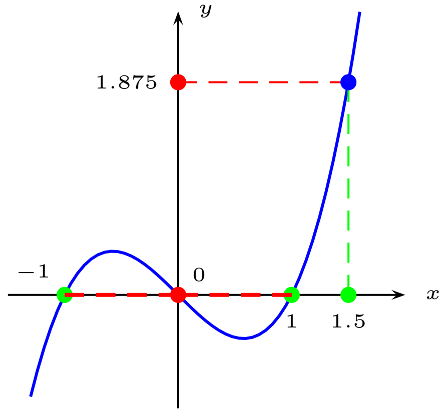
MATH1050 Handout: Image Sets and pre-image sets under 'nice' real-valued functions of one real variable

1. **'Concrete' examples on image sets under a 'nice' function from $\mathbb{R}$ to $\mathbb{R}$.**

   Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x^3 - x$ for any $x \in \mathbb{R}$.

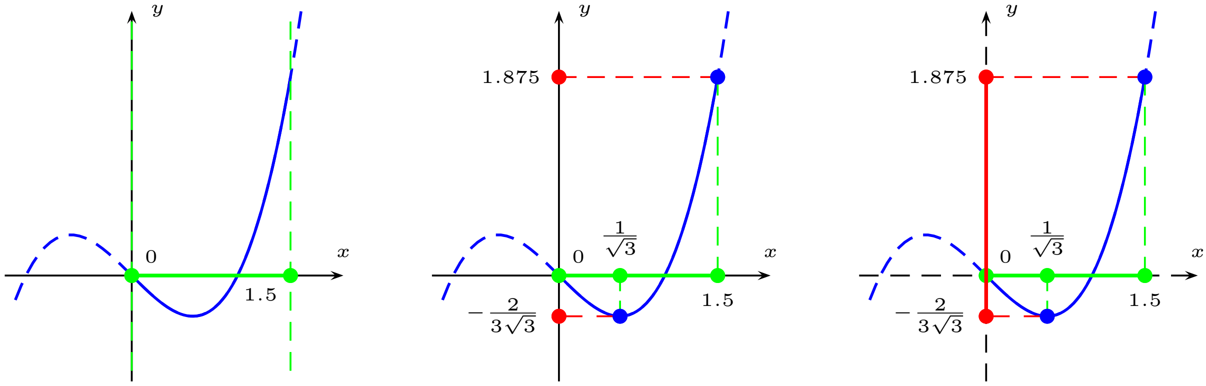
   (a)
   - What are $f(\{-1\})$, $f(\{-1,1\})$, $f(\{-1,1,1.5\})$?
   - How to read off the answer using the 'blobs-and-arrows diagram'?
   - How to read off the answer using the 'coordinate diagram'?





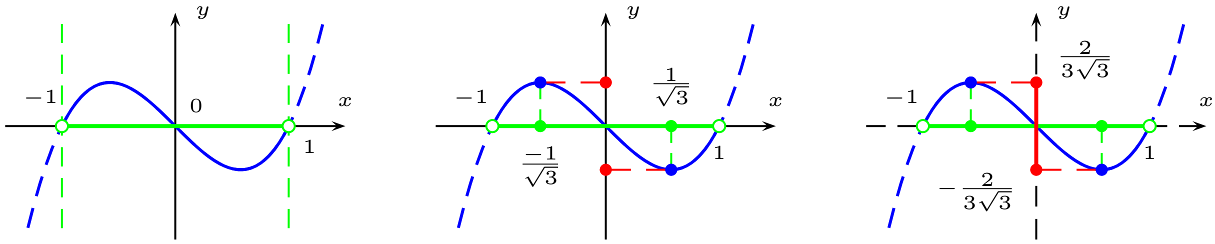
   $f(\{-1\}) = \{0\}$, $f(\{-1,1\}) = \{0\}$, $f(\{-1,1,1.5\}) = \{0, 1.875\}$.

   (b)
   - What is $f([0,1.5])$?
   - How to read off the answer using the 'coordinate diagram'?



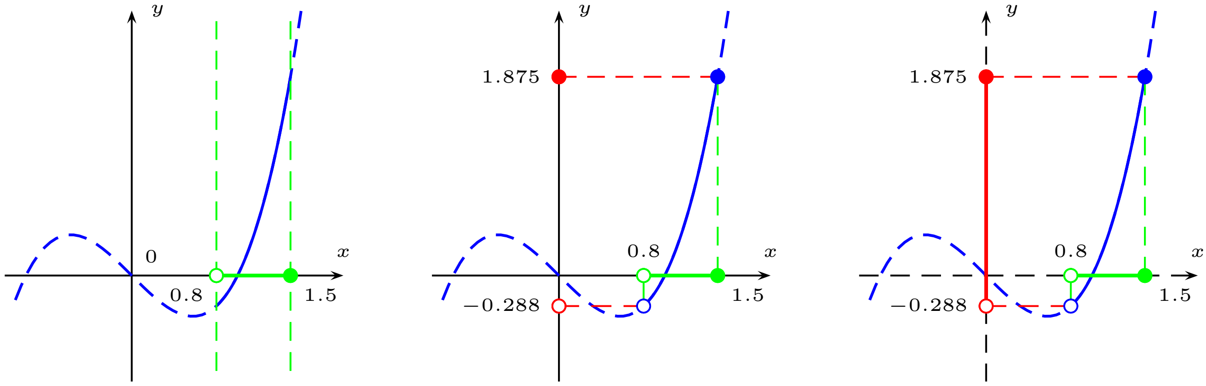
   $f([0,1.5]) = \left[\frac{-2}{3\sqrt{3}}, 1.875\right]$.

   (c)
   - What is $f((-1,1))$?
   - How to read off the answer using the 'coordinate diagram'?



   $f((-1,1)) = \left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right]$.

(d)
- What is $f((0.8, 1.5])$?
- How to read off the answer using the 'coordinate diagram'?



$f((0.8, 1.5]) = (-0.288, 1.875]$.

(e) How to prove, say, $f([0, 1.5]) = \left[\frac{-2}{3\sqrt{3}}, 1.875\right]$?

First ask: what to prove?
This is a set equality.

Then ask: what to do to prove such a set equality?
Prove both (†), (‡) below:

(†) *For any y, if $y \in f([0, 1.5])$ then $y \in \left[\frac{-2}{3\sqrt{3}}, 1.875\right]$.* [This is straightforward to verify.]

(‡) *For any y, if $y \in \left[\frac{-2}{3\sqrt{3}}, 1.875\right]$ then $y \in f([0, 1.5])$.* [This is difficult, but we can make use of the Intermediate Value Theorem, in light of the fact that f is continuous on $\mathbb{R}$.]

When appropriate we will freely use the continuity of the function f.
[Preparation. Check that f is continuous on $[0, 1.5]$. Also, apply whatever you know (such as, from one-variable calculus) to show that f attains on $[0, 1.5]$ the maximum at 1.5 and the minimum at $\frac{1}{\sqrt{3}}$.]

Argument for (†).

Pick any y. Suppose $y \in f([0, 1.5])$. There exists some $x \in [0, 1.5]$ such that $y = f(x)$.
[*Objective.* We want to deduce that for this same x, we have $\frac{-2}{3\sqrt{3}} \leq f(x) \leq 1.875$.]
Note that f is strictly decreasing on the interval $\left[0, \frac{1}{\sqrt{3}}\right]$, and f is strictly increasing on the interval $\left[\frac{1}{\sqrt{3}}, 1.5\right]$.
By continuity, f attains absolute minimum on $[0, 1.5]$ at $\frac{1}{\sqrt{3}}$.
By continuity, f attains absolute maximum on $[0, 1.5]$ at 0 or at 1.5. Since $f(0) = 0 < 1.875 = f(1.5)$, f attains absolute maximum on $[0, 1.5]$ at 1.5.
Now it follows that $-\frac{2}{3\sqrt{3}} = f(\frac{1}{\sqrt{3}}) \leq f(x) \leq f(1.5) = 1.875$.
Therefore $y \in \left[\frac{-2}{3\sqrt{3}}, 1.875\right]$.

Argument for (‡).

Pick any y. Suppose $y \in \left[\frac{-2}{3\sqrt{3}}, 1.875\right]$.
[*Objective.* For this same y, we want to name an appropriate $x \in [0, 1.5]$ which satisfies $y = f(x)$. So we want to solve the equation $y = f(u)$ with unknown u in $[0, 1.5]$.]
Note that $f(\frac{1}{\sqrt{3}}) = -\frac{2}{3\sqrt{3}}$ and $f(1.5) = 1.875$.
By the Intermediate-Value Theorem, there exists some $x \in \left[\frac{1}{\sqrt{3}}, 1.5\right]$ such that $f(x) = y$.
Note that $x \in [0, 1.5]$. Then $y \in f([0, 1.5])$.

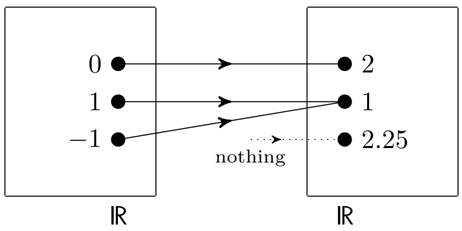
Remark. This is the statement of the **Intermediate Value Theorem**:

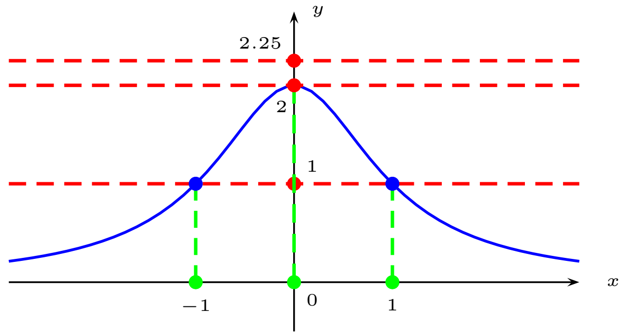
Let $a, b \in \mathbb{R}$, with $a < b$. Let $h : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose $h(a) \neq h(b)$. Suppose h is continuous on $[a, b]$. Then, for any $\gamma \in \mathbb{R}$, if γ is strictly between $h(a)$ and $h(b)$ then there exists some $c \in (a, b)$ such that $h(c) = \gamma$.

2. **'Concrete' examples on pre-image sets under a 'nice' function from $\mathbb{R}$ to $\mathbb{R}$.**

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = \dfrac{2}{x^2 + 1}$ for any $x \in \mathbb{R}$.

(a)
- What are $f^{-1}(\{2\})$, $f^{-1}(\{1\})$, $f^{-1}(\{2.25\})$?
- How to read off the answer using the 'blobs-and-arrows diagram'?
- How to interpret what we do in terms of solving equations?
- How to read off the answer using the 'coordinate diagram'?





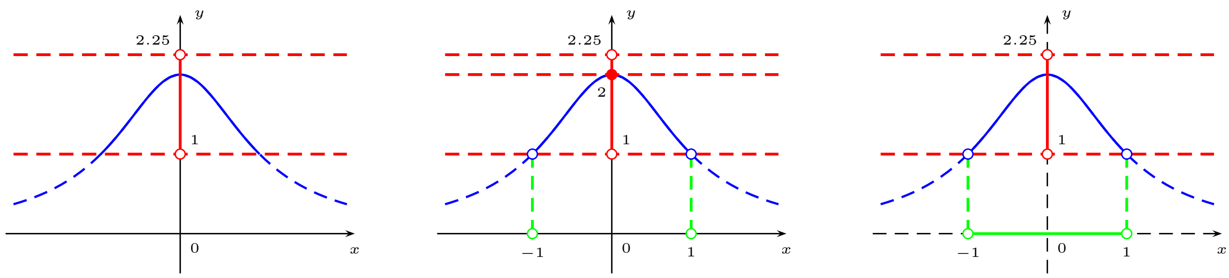
$f^{-1}(\{2\}) = \{0\}$, $f^{-1}(\{1\}) = \{-1, 1\}$, $f^{-1}(\{2.25\}) = \emptyset$.

Reminders.

(1) In general, the pre-image set of a non-empty set needs not be non-empty.

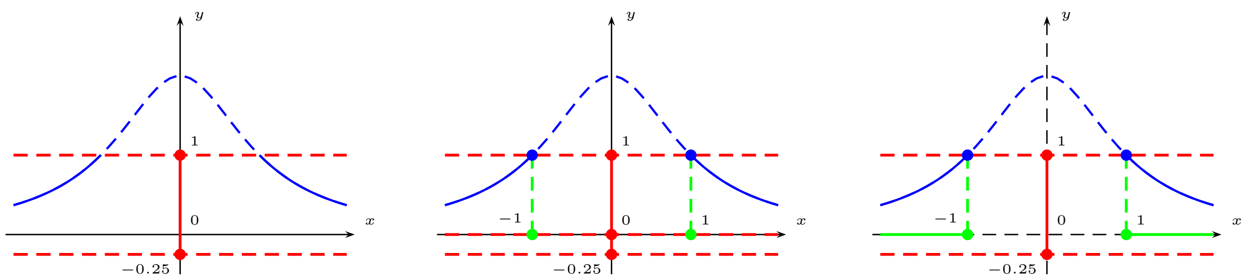
(2) In general, the pre-image set of a singleton needs not be a singleton.

(b)
- What is $f^{-1}((1, 2.25))$?
- How to read off the answer using the 'coordinate diagram'?
- How to interpret what we do in terms of solving equations/inequalities?



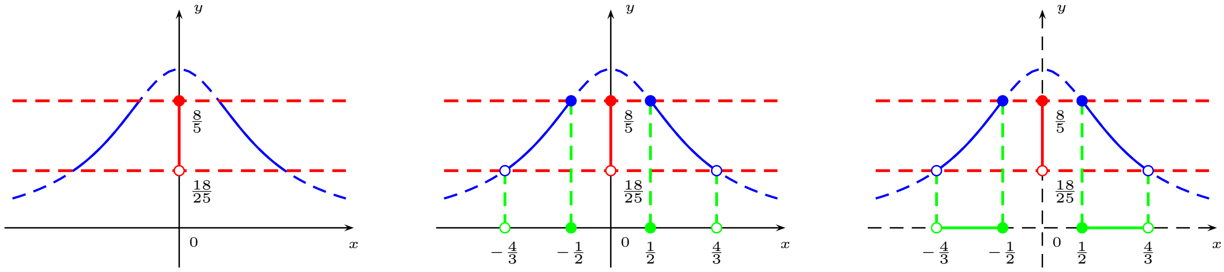
$f^{-1}((1, 2.25)) = (-1, 1)$.

(c)
- What is $f^{-1}([-0.25, 1])$?
- How to read off the answer using the 'coordinate diagram'?
- How to interpret what we do in terms of solving equations/inequalies?



$f^{-1}([-0.25, 1]) = (-\infty, -1] \cup [1, +\infty)$.

(d)
- What is $f^{-1}\left(\left(\frac{18}{25}, \frac{8}{5}\right]\right)$?
- How to read off the answer using the 'coordinate diagram'?
- How to interpret what we do in terms of solving equations/inequalities?



$$f^{-1}\left(\left(\frac{18}{25}, \frac{8}{5}\right]\right) = \left(-\frac{4}{3}, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{4}{3}\right).$$

(e) How to prove, say, $f^{-1}((1, 2.25)) = (-1, 1)$?

First ask: what to prove?
This is a set equality.

Then ask: what to do to prove such a set equality?
Prove both (†), (‡) below:

(†) *For any x, if $x \in f^{-1}((1, 2.25))$ then $x \in (-1, 1)$.*
(‡) *For any x, if $x \in (-1, 1)$ then $x \in f^{-1}((1, 2.25))$.*

Argument for (†).

Pick any x. Suppose $x \in f^{-1}((1, 2.25))$.
[*Objective.* We want to deduce that $x \in (-1, 1)$.]
There exists some $y \in (1, 2.25)$ such that $y = f(x)$.
For the same x, y, we have

$$\begin{cases} \dfrac{2}{1+x^2} = f(x) = y < 2.25 \\ \dfrac{2}{1+x^2} = f(x) = y > 1 \end{cases}$$

[The inequality $\dfrac{2}{1+x^2} < 2.25$ is not useful.]
Since $1 < \dfrac{2}{1+x^2}$, we have $1 + x^2 < 2$. Then $-1 < x < 1$. Therefore $x \in (-1, 1)$.

Argument for (‡).

Pick any x. Suppose $x \in (-1, 1)$.
[*Objective.* We want to deduce that there exists some $y \in (1, 2.25)$ such that $y = f(x)$.]
Take $y = f(x)$. [We want to deduce $1 < y < 2.25$. Ask: What does '$y = f(x)$' give? This gives '$y = \dfrac{2}{1+x^2}$'.]
We have $-1 < x < 1$. Then $x^2 < 1$. Therefore $1 \leq 1 + x^2 < 2$.
Since $0 < 1 + x^2 < 2$, we have $y = f(x) = \dfrac{2}{1+x^2} > 1$.
Since $1 + x^2 \geq 1$, we have $y = f(x) = \dfrac{2}{1+x^2} \leq 2 < 2.25$.
Therefore $1 < y < 2.25$. Hence $y \in (1, 2.25)$.
Hence $x \in f^{-1}((1, 2.25))$.