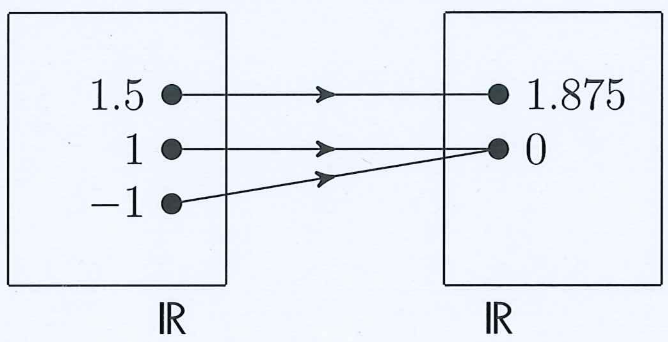
1. **'Concrete' examples on image sets under a 'nice' function from $\mathbb{R}$ to $\mathbb{R}$.**

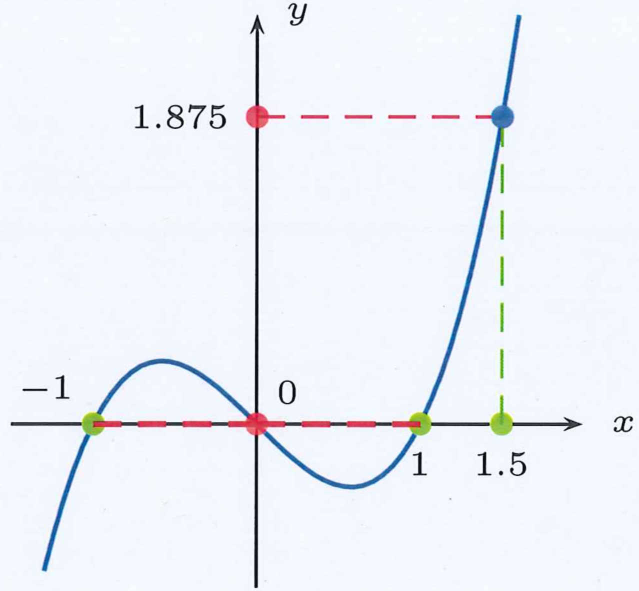
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x^3 - x$ for any $x \in \mathbb{R}$.

(a) • What are $f(\{-1\})$, $f(\{-1, 1\})$, $f(\{-1, 1, 1.5\})$?

• How to read off the answer using the 'blobs-and-arrows diagram'?

• How to read off the answer using the 'coordinate diagram'?

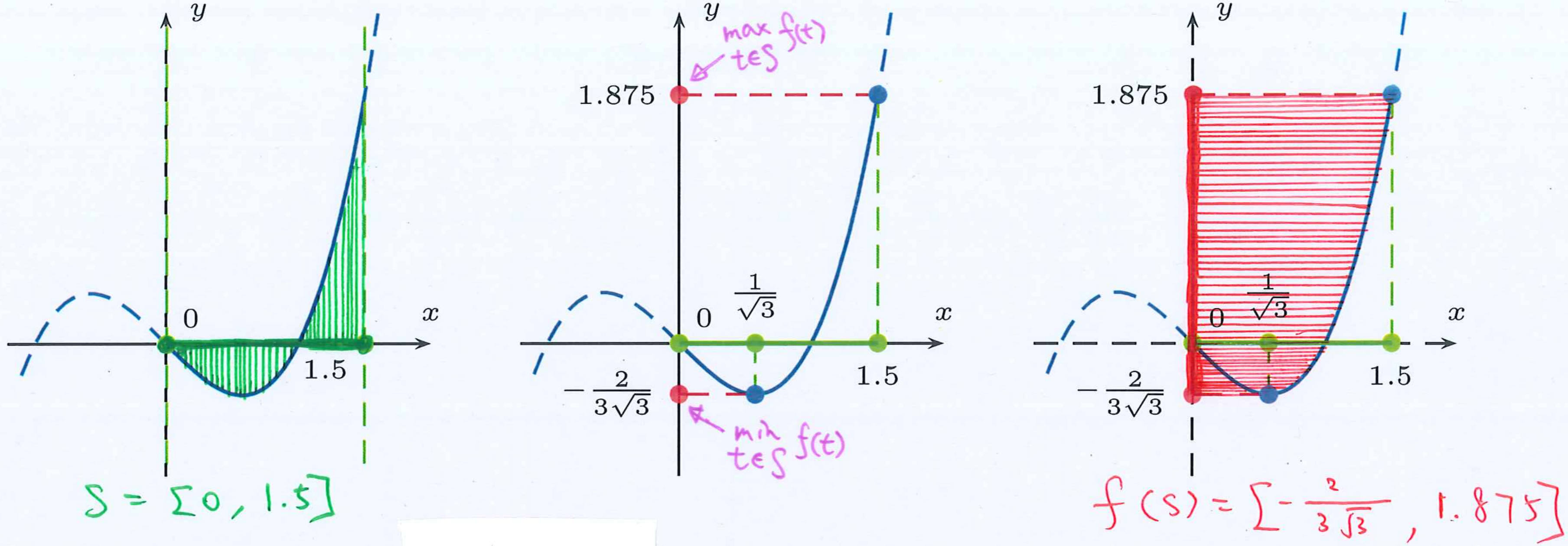




S	$f(S)$
$\{-1\}$	$\{f(-1)\} = \{0\}$
$\{-1, 1\}$	$\{f(-1), f(1)\} = \{0\}$
$\{-1, 1, 1.5\}$	$\{f(-1), f(1), f(1.5)\} = \{0, 1.875\}$

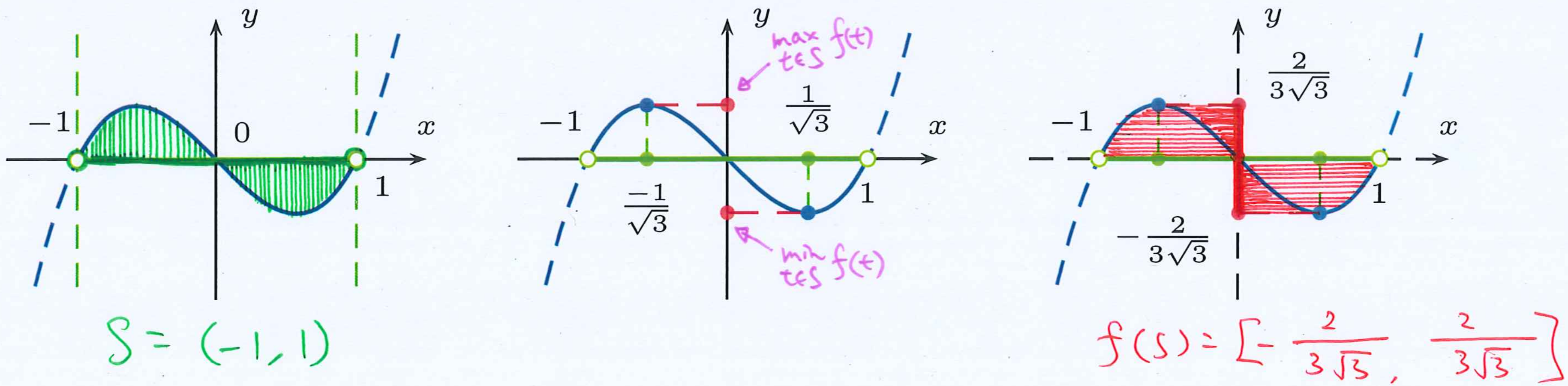
(b) • What is $f([0, 1.5])$?

• How to read off the answer using the 'coordinate diagram'?



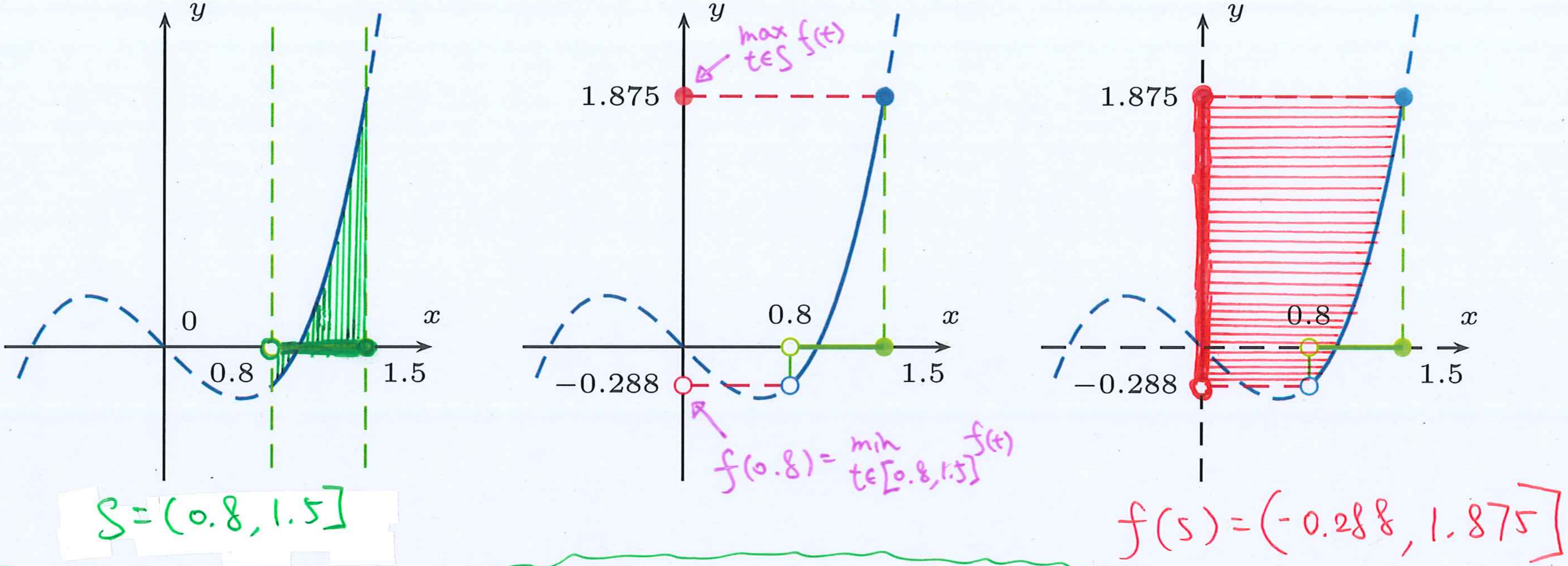
(c) • What is $f((-1, 1))$?

• How to read off the answer using the 'coordinate diagram'?



(d) • What is $f((0.8, 1.5])$?

• How to read off the answer using the 'coordinate diagram'?



We expect the image sets of intervals under f to be intervals, and the image sets of closed and bounded intervals under f to be closed and bounded intervals.
This is guaranteed by the theory of continuity.

Theorem (A).
Let D be a subset of $\mathbb{R}$, and $f: D \to \mathbb{R}$ be a function.
Suppose I is an interval, lying entirely inside D. Suppose f is continuous on I.
Then the statements below hold:
(1) $f(I)$ is an interval. (2) If I is closed and bounded, then $f(I)$ is closed and bounded.

(e) How to prove, say, $f([0,1.5]) = \left[\frac{-2}{3\sqrt{3}}, 1.875\right]$?

First ask: what to prove?

This is a set equality.

Then ask: what to do to prove such a set equality?

Prove both (†), (‡) below:

(†) *For any y, if $y \in f([0,1.5])$ then $y \in \left[\frac{-2}{3\sqrt{3}}, 1.875\right]$.*

(‡) *For any y, if $y \in \left[\frac{-2}{3\sqrt{3}}, 1.875\right]$ then $y \in f([0,1.5])$.*

When appropriate we will freely use the continuity of the function f.

How to prove (†)?

(†) reads: '*For any* y, *if* $y \in f([0, 1.5])$ *then* $y \in \left[\frac{-2}{3\sqrt{3}}, 1.875\right]$.'

Inspect (†) carefully:

- '$y \in f([0, 1.5])$' reads: there exists some $x \in [0, 1.5]$ such that $y = f(x)$.
- '$y \in \left[\frac{-2}{3\sqrt{3}}, 1.875\right]$' reads: $\frac{-2}{3\sqrt{3}} \leq y \leq 1.875$.

Argument for (†):

Pick any $y \in f([0, 1.5])$.

There exists some $x \in [0, 1.5]$ such that $y = f(x)$. [We want to deduce $\frac{-2}{3\sqrt{3}} \leq y \leq 1.875$.]

Note that f is continuous on $[0, 1.5]$.

Also, $\begin{cases} f \text{ is strictly decreasing on } \left[0, \frac{1}{\sqrt{3}}\right]. \\ f \text{ is strictly increasing on } \left[\frac{1}{\sqrt{3}}, 1.5\right]. \end{cases}$

Moreover, $f(0) = 0$, $f\left(\frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$, $f(1.5) = 1.875$.

So $-\frac{2}{3\sqrt{3}} = \min_{t \in [0,1.5]} f(t) \leq y = f(x) \leq \max_{t \in [0,1.5]} f(t) = 1.875$.

Hence $y \in \left[-\frac{2}{3\sqrt{3}}, 1.875\right]$. $\square$

How to prove (‡)?

(‡) reads: '*For any* y, *if* $y \in \left[\dfrac{-2}{3\sqrt{3}}, 1.875\right]$ *then* $y \in f([0, 1.5])$.'

Inspect (‡) carefully:

- '$y \in \left[\dfrac{-2}{3\sqrt{3}}, 1.875\right]$' reads: $\dfrac{-2}{3\sqrt{3}} \leq y \leq 1.875$.
- '$y \in f([0, 1.5])$' reads: there exists some $x \in [0, 1.5]$ such that $y = f(x)$.

Argument for (‡):

Pick any $y \in \left[\dfrac{-2}{3\sqrt{3}}, 1.875\right]$. By definition, $-\dfrac{2}{3\sqrt{3}} \leq y \leq 1.875$.

[*Objective.* For this same y, we want to name an appropriate $x \in [0, 1.5]$ which satisfies $y = f(x)$. So we want to solve the equation $y = f(u)$ with unknown u in $[0, 1.5]$.]

Note that $f\left(\dfrac{1}{\sqrt{3}}\right) = -\dfrac{2}{3\sqrt{3}}$ and $f(1.5) = 1.875$.

Also, $\left[\dfrac{1}{\sqrt{3}}, 1.5\right] \subset [0, 1.5]$. Moreover, f is continuous on $\left[\dfrac{1}{\sqrt{3}}, 1.5\right]$.

By the Intermediate Value Theorem, there exists some $x \in \left[\frac{1}{\sqrt{3}}, 1.5\right]$ such that $y = f(x)$.

By definition, $x \in [0, 1.5]$ and $y = f(x)$. Hence $y \in f([0, 1.5])$. $\square$

Remark.

This is the statement of the **Intermediate Value Theorem**:

Let $a, b \in \mathbb{R}$, with $a < b$.
Let $h : [a, b] \longrightarrow \mathbb{R}$ be a function.
Suppose $h(a) \neq h(b)$.
Suppose h is continuous on $[a, b]$.
Then, for any $\gamma \in \mathbb{R}$, if γ is strictly between $h(a)$ and $h(b)$ then there exists some $c \in (a, b)$ such that $h(c) = \gamma$.

It is logically equivalent to **Bolzano's Intermediate Value Theorem**:

Let $a, b \in \mathbb{R}$, with $a < b$.
Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is a function which is continuous on $[a, b]$.
Suppose $f(a)f(b) < 0$.
Then there exists some $x_0 \in (a, b)$ such that $f(x_0) = 0$.

In the context of the statement of Bolzano's Intermediate Value Theorem, x_0 is called a **zero of f in** (a, b).

2. **'Concrete' examples on pre-image sets under a 'nice' function from $\mathbb{R}$ to $\mathbb{R}$.**

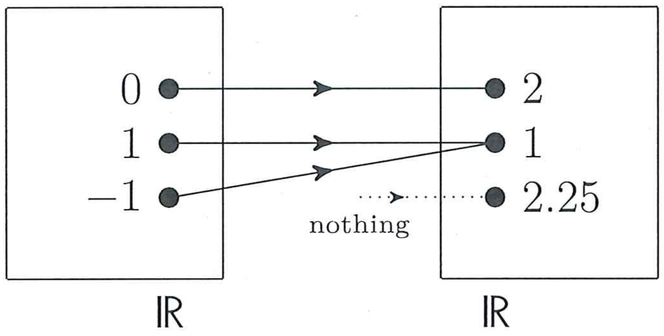
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = \dfrac{2}{x^2+1}$ for any $x \in \mathbb{R}$.

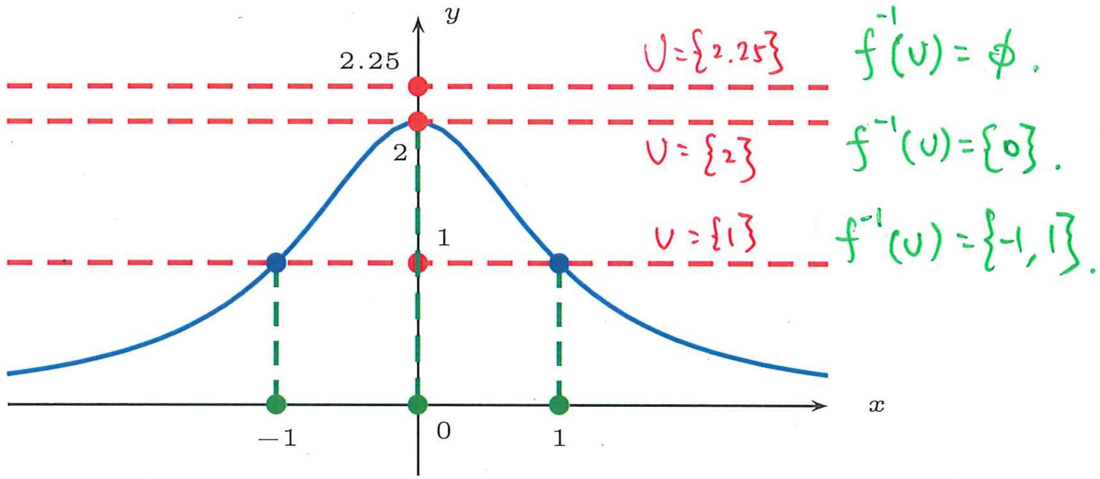
(a) • What are $f^{-1}(\{2\})$, $f^{-1}(\{1\})$, $f^{-1}(\{2.25\})$?

• How to read off the answer using the 'blobs-and-arrows diagram'?

• How to interpret what we do in terms of solving equations?

• How to read off the answer using the 'coordinate diagram'?





Reminders.

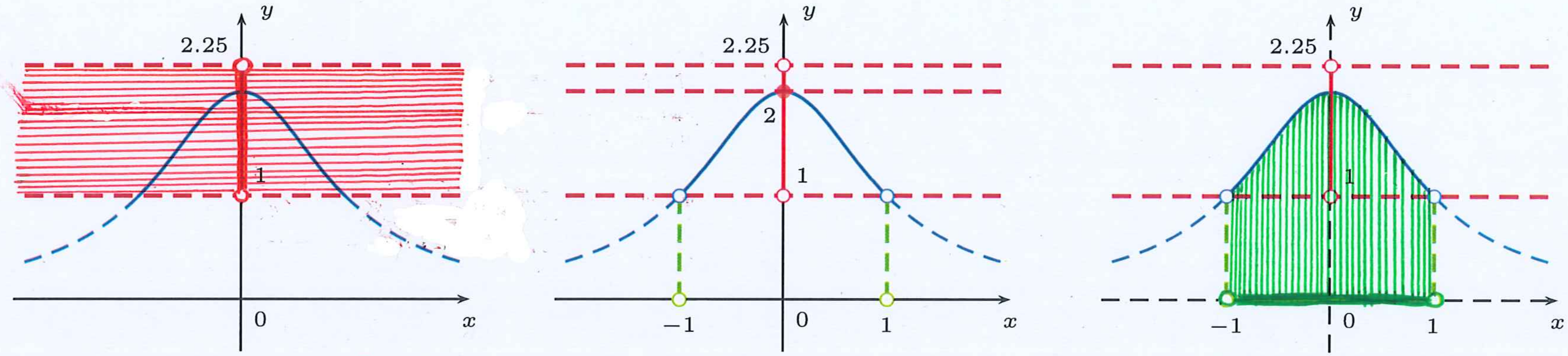
(1) In general, the pre-image set of a non-empty set needs not be non-empty.

(2) In general, the pre-image set of a singleton needs not be a singleton.

(b) • What is $f^{-1}((1, 2.25))$?

• How to read off the answer using the ‘coordinate diagram’?

• How to interpret what we do in terms of solving equations/inequalities?

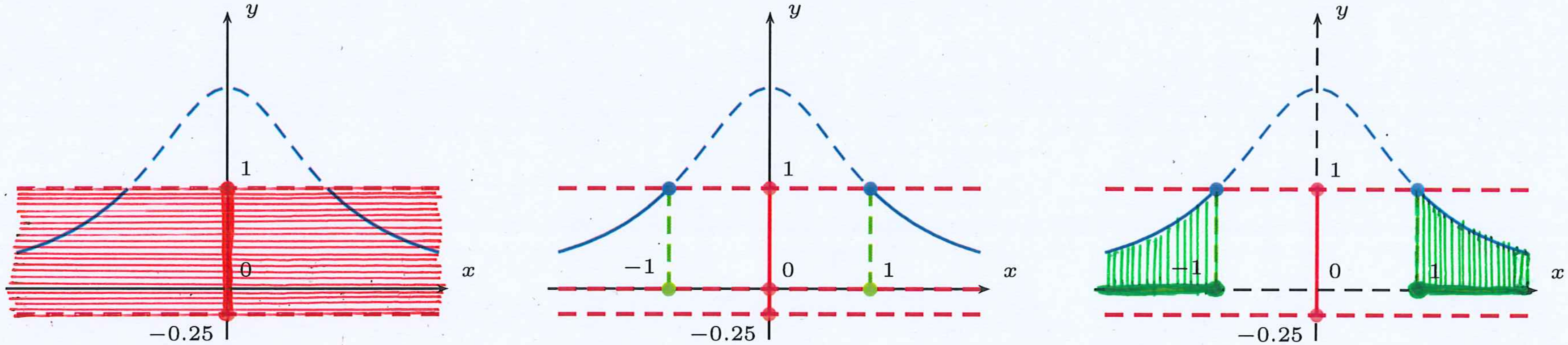


$U = (1, 2.25)$ $\qquad$ $f^{-1}(U) = (-1, 1)$

(c) • What is $f^{-1}([-0.25, 1])$?

• How to read off the answer using the ‘coordinate diagram’?

• How to interpret what we do in terms of solving equations/inequalies?

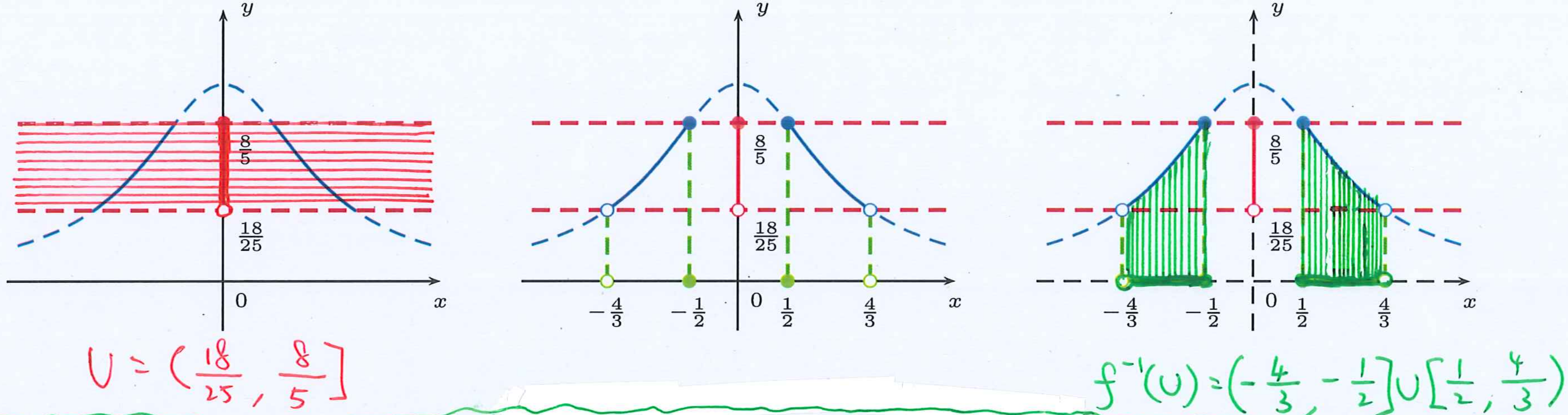


$U = [-0.25, 1]$ $\qquad$ $f^{-1}(U) = (-\infty, -1] \cup [1, +\infty)$

(d) • What is $f^{-1}\left(\left(\frac{18}{25}, \frac{8}{5}\right]\right)$?

• How to read off the answer using the 'coordinate diagram'?

• How to interpret what we do in terms of solving equations/inequalities?



We expect the pre-image sets of open intervals under f to be open intervals, or unions of open intervals.

This is guaranteed by the theory of continuity.

Theorem (B).

Let D be a subset of $\mathbb{R}$, and $f : D \to \mathbb{R}$ be a function.

Suppose f is continuous on D. Suppose J is an open interval.

Then there exists some subset S of $\mathbb{R}$ such that S is open in $\mathbb{R}$ and $f^{-1}(J) = S \cap D$.

(Such a set S is open in $\mathbb{R}$ in the sense that S is a (generalised) union of open intervals.)

(e) How to prove, say, $f^{-1}((1, 2.25)) = (-1, 1)$?

First ask: what to prove?

This is a set equality.

Then ask: what to do to prove such a set equality?

Prove both (†), (‡) below:

(†) *For any x, if $x \in f^{-1}((1, 2.25))$ then $x \in (-1, 1)$.*

(‡) *For any x, if $x \in (-1, 1)$ then $x \in f^{-1}((1, 2.25))$.*

How to prove (†)?

(†) reads: '*For any x, if $x \in f^{-1}((1, 2.25))$ then $x \in (-1, 1)$.*'

Inspect (†) carefully:

- '$x \in f^{-1}((1, 2.25))$' reads: there exists some $y \in (1, 2.25)$ such that $y = f(x)$.

- '$x \in (-1, 1)$' reads: $-1 < x < 1$.

Argument for (†)*:*

Pick any x. Suppose $x \in f^{-1}((1, 2.25))$. Then there exists some $y \in (1, 2.25)$ such that $y = f(x)$.

[*Objective.* We want to deduce that $x \in (-1, 1)$.]

Note that, for the same x and y, we have

$$\begin{cases} \dfrac{2}{1+x^2} = f(x) = y < 2.25 \\ \dfrac{2}{1+x^2} = f(x) = y > 1 . \end{cases}$$

- Since $\dfrac{2}{1+x^2} < 2.25$, we have $x^2 > \dfrac{2}{2.25} - 1 = -\dfrac{1}{9}$. [This is not useful.]
- Since $\dfrac{2}{1+x^2} > 1$, we have $x^2 < \dfrac{2}{1} - 1 = 1$. Then $-1 < x < 1$.

Hence $x \in (-1, 1)$. $\square$

How to prove (‡)?

(‡) reads: '*For any* x, *if* $x \in (-1, 1)$ *then* $x \in f^{-1}((1, 2.25))$.'

Inspect (‡) carefully:

- '$x \in (-1, 1)$' reads: $-1 < x < 1$.
- '$x \in f^{-1}((1, 2.25))$' reads: there exists some $y \in (1, 2.25)$ such that $y = f(x)$.

Argument for (‡):

Pick any x. Suppose $x \in (-1, 1)$. Then $-1 < x < 1$.

[*Objective.* We want to deduce that there exists some $y \in (1, 2.25)$ such that $y = f(x)$.]

Take $y = f(x)$. [We want to deduce $1 < y < 2.25$.
So ask: What does '$y = f(x)$' give? It gives '$y = \frac{2}{1+x^2}$'.]

Since $-1 < x < 1$, we have $x^2 < 1$.

Then $0 < x^2 + 1 < 2$.

Therefore $1 < \frac{2}{x^2+1} = f(x) = y$.

Also note that $y = f(x) = \frac{2}{x^2+1} \leq \frac{2}{0^2+1} = 2 < 2.25$.

Therefore $1 < y < 2.25$. It follows that $x \in f^{-1}((1, 2.25))$. $\square$