

1. Definition.

Let A, B be sets and $f : A \rightarrow B$ be a function from A to B .

Let S be a subset of A .

The **image set** of the set S under the function f is defined to be the set

$$\{y \in B : \text{There exists some } x \in S \text{ such that } y = f(x)\}$$

(or $\{y \in B : y = f(x) \text{ for some } x \in S\}$, or $\{f(x) \mid x \in S\}$).

It is denoted by $f(S)$.

$Q(x, y)$ is ' $y = f(x)$ '.

$$(\exists x \in S) Q(x, y)$$

↑
This is a predicate
with variable y .

Remarks.

(1) **Terminology.** We write $f(A) = \{y \in B : y = f(x) \text{ for some } x \in A\}$.

The set $f(A)$ is called the **image of the function** f .

(2) **Warning.** ' $f(S)$ ' should be understood as one symbol which is the name of some subset of B whose 'content' depends on f and S , according to the predicate

'there exists some $x \in S$ such that $y = f(x)$ '

for its construction via the Method of Specification.

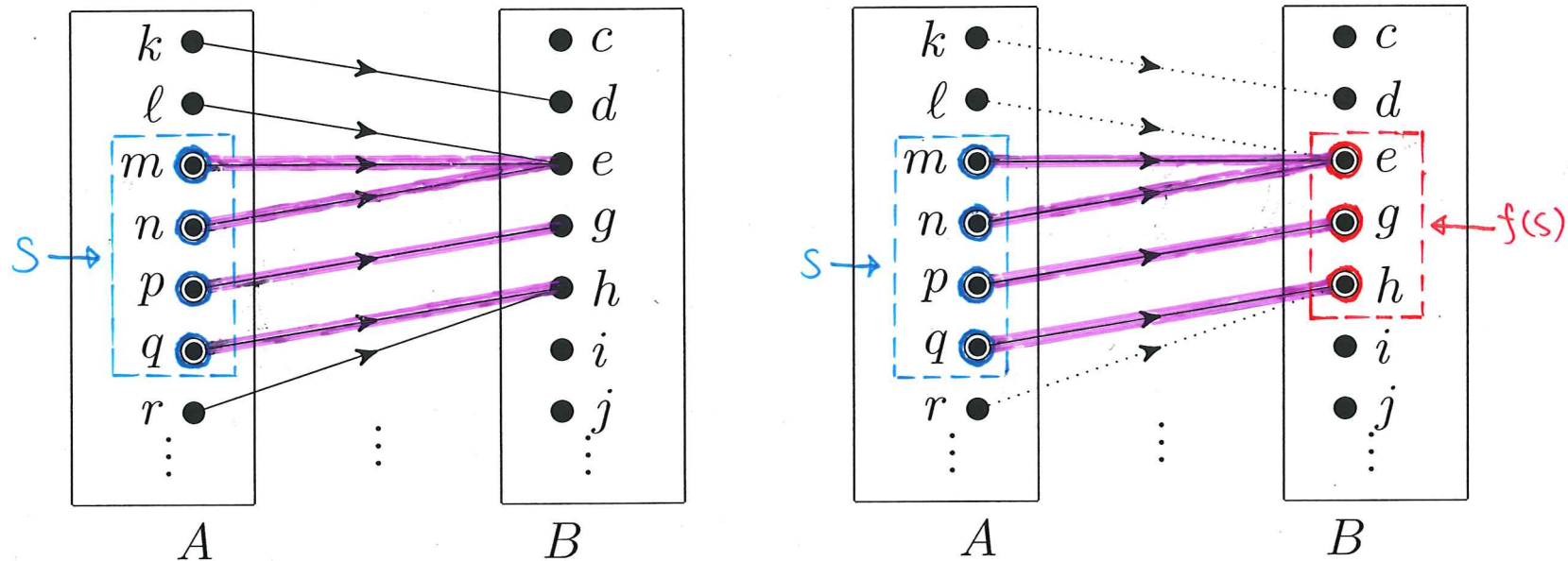
2. Example of pictorial visualizations of image sets.

Let $A = \{k, \ell, m, n, p, q, r, \dots\}$, $B = \{c, d, e, g, h, i, j, \dots\}$, and $f : A \rightarrow B$ be the function defined by $f(k) = d$, $f(\ell) = f(m) = f(n) = e$, $f(p) = g$, $f(q) = f(r) = h$, Let $S = \{m, n, p, q\}$. We have $f(S) = \{f(m), f(n), f(p), f(q)\} = \{e, g, h\}$.

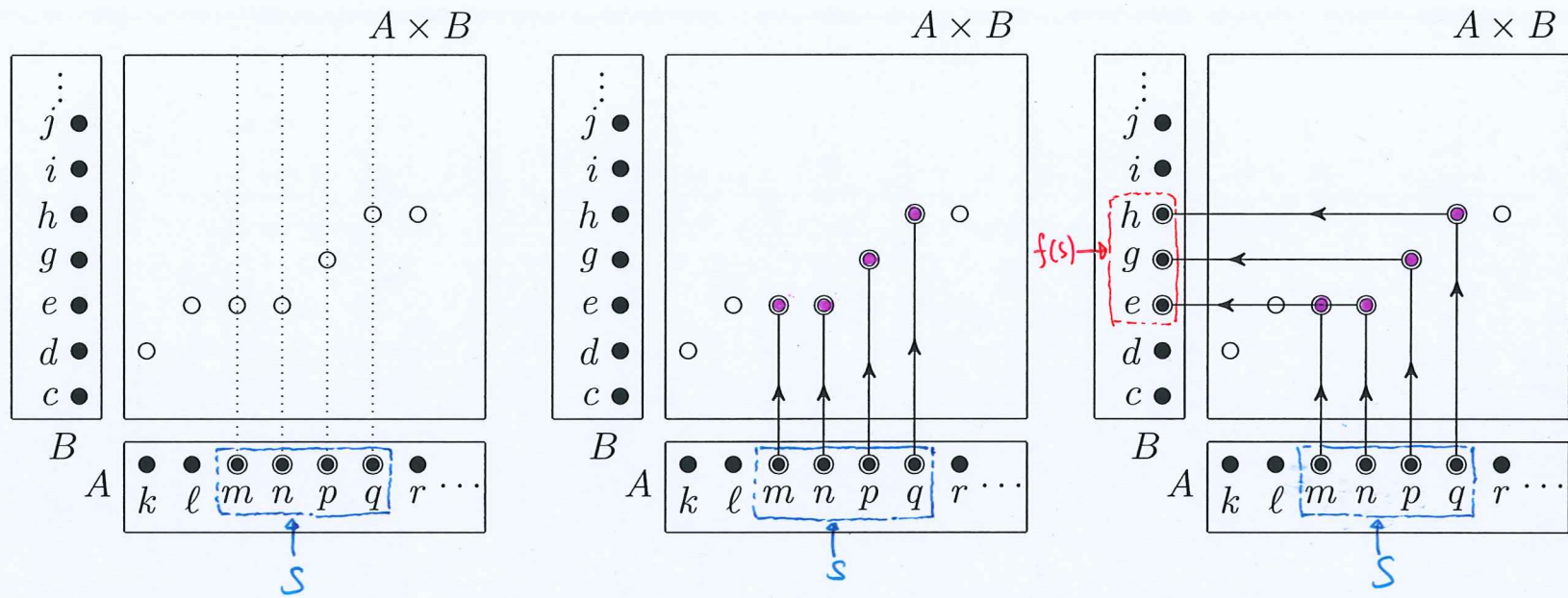
(a) 'Blobs-and-arrows diagram'.

Two equivalent ways to see what $f(S)$ is:

- (i) Examine each $y \in B$. Ask if it is targetted by some $x \in S$ via f . If yes, collect it; if no, throw away.
- (ii) Simply collect the target of every $x \in S$.



(b) 'Coordinate plane diagram'.



3. Definition.

Let A, B be sets and $f : A \rightarrow B$ be a function from A to B .

Let U be a subset of B .

The **pre-image set** of the set U under the function f is defined to be the set

$$\{x \in A : \text{There exists some } y \in U \text{ such that } y = f(x)\}$$

(or $\{x \in A : y = f(x) \text{ for some } y \in U\}$, or $\{x \in A : f(x) \in U\}$).

It is denoted by $f^{-1}(U)$.

$Q(x, y)$ is ' $y = f(x)$ '.

$$(\exists y \in U) Q(x, y)$$

↑
This is a predicate
with variable x .

Warnings.

- (1) ' $f^{-1}(U)$ ' should be understood as one symbol which is the name of some subset of A whose 'content' depends on f and U , according to the predicate

'there exists some $y \in U$ such that $y = f(x)$ '

for its construction via the Method of Specification.

- (2) The presence of the chain of symbols ' f^{-1} ' in ' $f^{-1}(U)$ ' does not give any hint as to whether f is 'invertible'/bijective as a function or not.

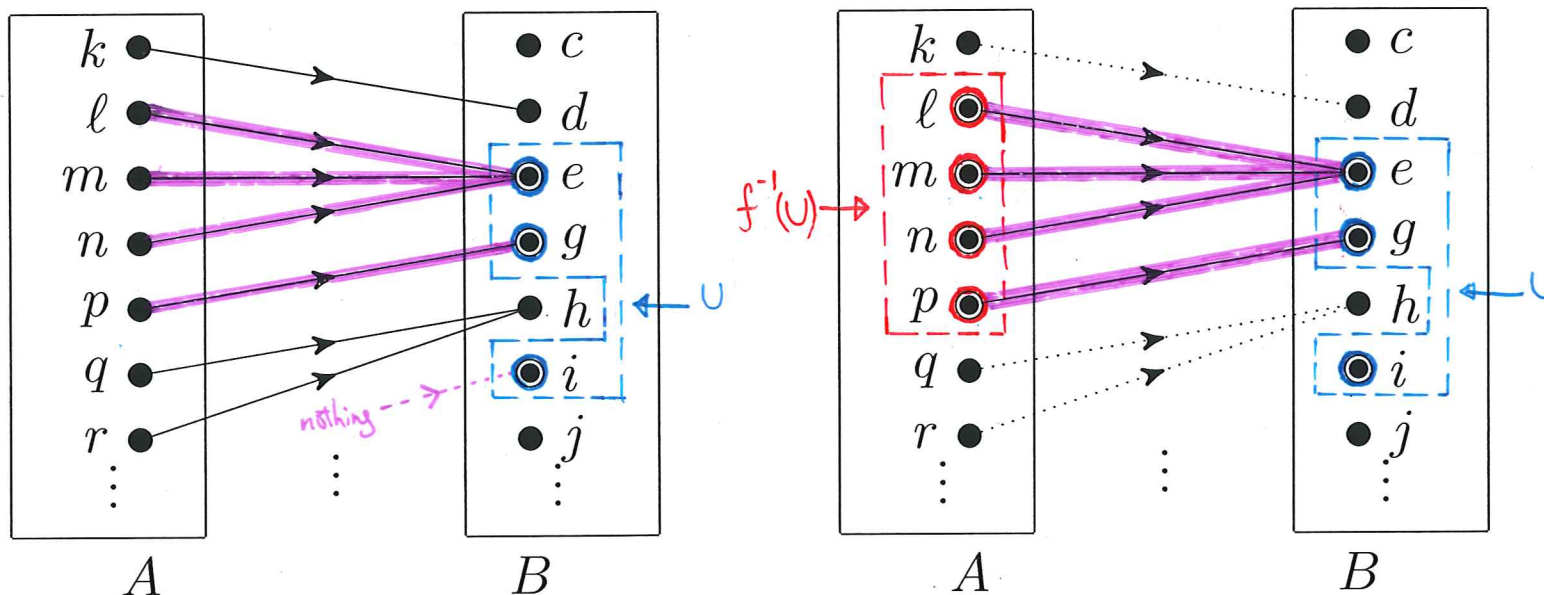
4. Example of pictorial visualizations of pre-image sets.

Let $A = \{k, \ell, m, n, p, q, r, \dots\}$, $B = \{c, d, e, g, h, i, j, \dots\}$, and $f : A \rightarrow B$ be the function defined by $f(k) = d$, $f(\ell) = f(m) = f(n) = e$, $f(p) = g$, $f(q) = f(r) = h$, Let $U = \{e, g, i\}$. We have $f^{-1}(U) = \{\ell, m, n, p\} = \{\ell, m, n\} \cup \{p\} \cup \emptyset = f^{-1}(\{e\}) \cup f^{-1}(\{g\}) \cup f^{-1}(\{i\})$.

(a) 'Blobs-and-arrows diagram'.

Two equivalent ways to see what $f^{-1}(U)$ is:

- (i) Examine each $x \in A$. Ask if it targets some $y \in U$ via f . If yes, collect it; if no, throw away.
- (ii) For each $y \in U$, collect whatever in A that targets it, and then 'take union'.



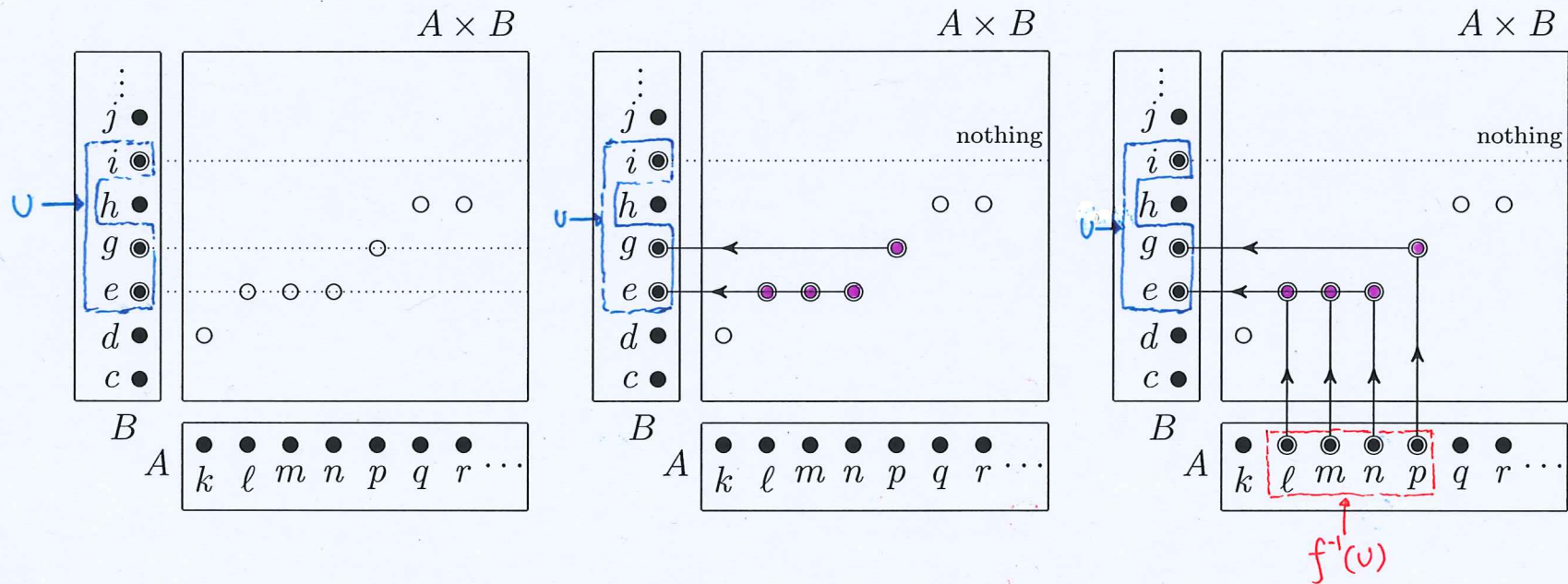
(b) 'Coordinate plane diagram'.

For each $y_0 \in B$, to determine the pre-image set $f^{-1}(\{y_0\})$ of the singleton $\{y_0\}$ under f is the same as:

- to determine the **solution set** of the equation $y_0 = f(x)$ with unknown x in A .

We call this set the **level set** at y_0 of the function f .

So $f^{-1}(U)$ is the 'union' of the level sets of the y 's in U of the function f .



Solution set of the equation $e=f(u)$ with unknown u in A ? $\{l, m, n\}$
 Solution set of the equation $g=f(u)$ with unknown u in A ? $\{p\}$
 Solution set of the equation $i=f(u)$ with unknown u in A ? \emptyset