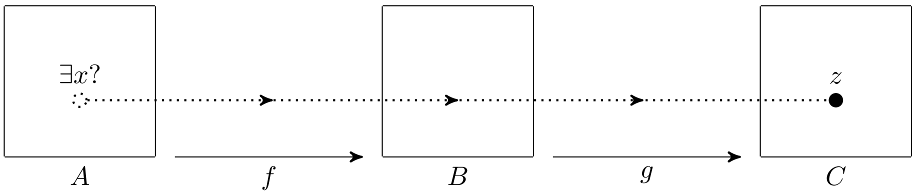
MATH1050 Compositions, Surjectivity and Injectivity

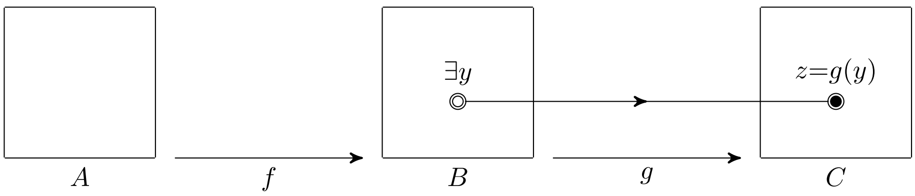
1. **Definitions.**
   (a) *Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Define the function $g \circ f : A \longrightarrow C$ by $(g \circ f)(x) = g(f(x))$ for any $x \in A$. $g \circ f$ is called the* **composition** *of the functions f, g.*
   (b) *Let D, R be sets, and $h : D \longrightarrow R$ be a function.*
      i. *h is said to be* **surjective** *if (for any $v \in R$ there exists some $u \in D$ such that $v = h(u)$).*
      ii. *h is said to be* **injective** *if (for any $t, u \in D$, if $h(t) = h(u)$ then $t = u$).*

2. **Theorem ($\sharp_1$).**
   Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. The following statements hold:
   (1) *Suppose f, g are surjective. Then $g \circ f$ is surjective.*
   (2) *Suppose f, g are injective. Then $g \circ f$ is injective.*

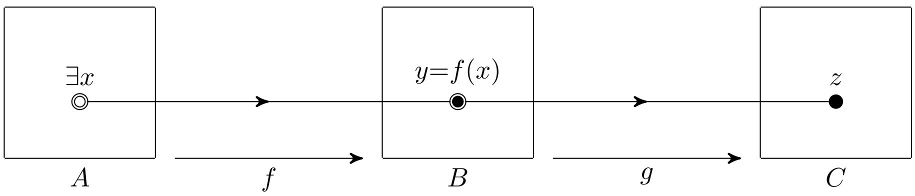
3. **Proof of Statement (1) of Theorem ($\sharp_1$) (with pictures).**
   Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.
   Suppose f, g are surjective. [We want to verify that $g \circ f$ is surjective under this assumption. By the definition of surjectivity, this is the same as verifying that for any $z \in C$, there exists some $x \in A$ such that $z = (g \circ f)(x)$.]
   (1) Pick any $z \in C$.
   [z is 'arbitrarily picked' from C. However, from this moment on, this z is fixed. The letter 'z' always refers this same element of C. We want to argue that there is some $x \in A$ satisfying $(g \circ f)(x) = z$.]



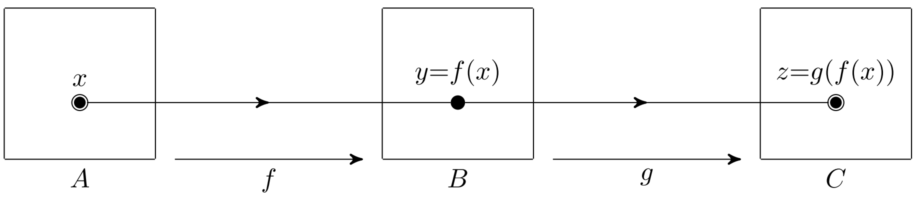
   (2) For this $z \in C$, by the surjectivity of g, there exists some $y \in B$ such that $z = g(y)$.
   [What is said here? (i) y is 'generated' by z. (ii) $y \in B$. (iii) $z = g(y)$.]



   (3) For the same $y \in B$, by the surjectivity of f, there exists some $x \in A$ such that $y = f(x)$.
   [What is said here? (i) x is 'generated' by y. (ii) $x \in A$. (iii) $y = f(x)$.]



   (4) For the same $z \in C$, $x \in A$, we have $z = g(f(x)) = (g \circ f)(x)$.



   It follows that $g \circ f$ is surjective.

Proof of Statement (1) of Theorem ($\sharp_1$) (without pictures).

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.

Suppose f, g are surjective. [We want to verify that $g \circ f$ is surjective under this assumption. By the definition of surjectivity, this is the same as verifying that for any $z \in C$, there exists some $x \in A$ such that $z = (g \circ f)(x)$.]

Pick any $z \in C$. [We want to argue that for this same z, there is some x satisfying $z = (g \circ f)(x)$.]
For this $z \in C$, by the surjectivity of g, there exists some $y \in B$ such that $z = g(y)$.
For the same $y \in B$, by the surjectivity of f, there exists some $x \in A$ such that $y = f(x)$.
For the same $z \in C$, $x \in A$, we have $z = g(f(x)) = (g \circ f)(x)$.

It follows that $g \circ f$ is surjective.

Very formal proof of Statement (1) of Theorem ($\sharp_1$).

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose f is surjective and g is surjective. [We want to verify that under the above assumption, $g \circ f$ is surjective.]

Pick any object z.

I. Suppose $z \in C$. [Assumption.]

II. g is surjective. [Assumption.]

III. There exists some $y \in B$ such that $z = g(y)$. [**I**, **II**, definition of surjectivity.]

IIIi. $y \in B$. [**III**.]

IIIii. $z = g(y)$. [**III**.]

IV. f is surjective. [Assumption.]

V. There exists some $x \in A$ such that $y = f(x)$. [**IIIi**, **IV**, definition of surjectivity.]

Vi. $x \in A$. [**V**.]

Vii. $y = f(x)$. [**V**.]

VII. $z = g(y)$ and $y = f(x)$. [**IIIii**, **Vii**]

VIII. $z = g(f(x))$. [**VII**]

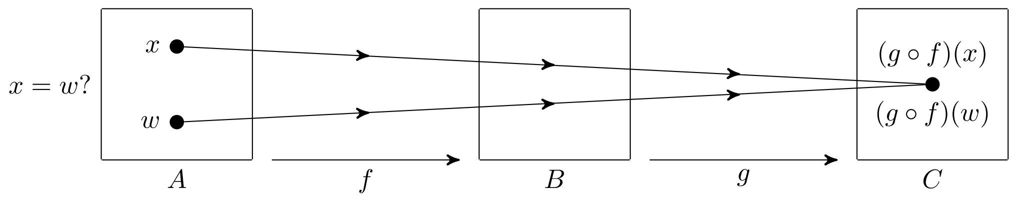
IX. $z = (g \circ f)(x)$. [Definition of composition.]

Hence $g \circ f$ is surjective.

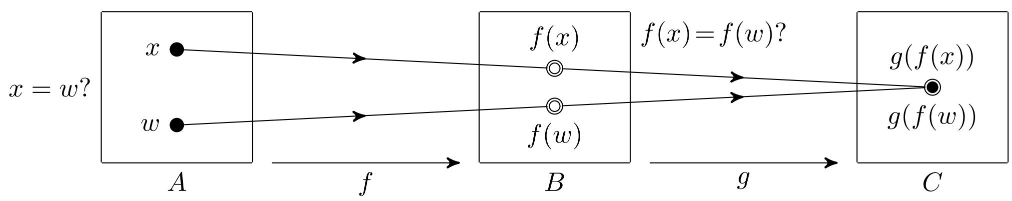
4. **Proof of Statement (2) of Theorem ($\sharp_1$) (with pictures).**

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose f, g are injective. [We want to verify that $g \circ f$ is injective under this assumption. By the definition of injectivity, this is the same as verify that for any $x, w \in A$, if $(g \circ f)(x) = (g \circ f)(w)$ then $x = w$.]

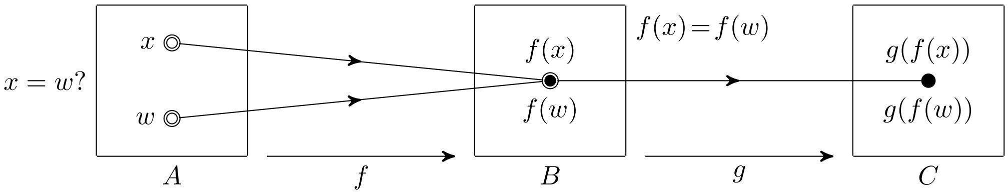
(1) Pick any $x, w \in A$. [x, w are 'arbitrarily picked' from A. However, from this moment on, these x, w are fixed. The letters 'x, w' always refer to these same element(s) of A. We want to argue that if $(g \circ f)(x) = (g \circ f)(w)$ then $x = w$.]
Suppose $(g \circ f)(x) = (g \circ f)(w)$.



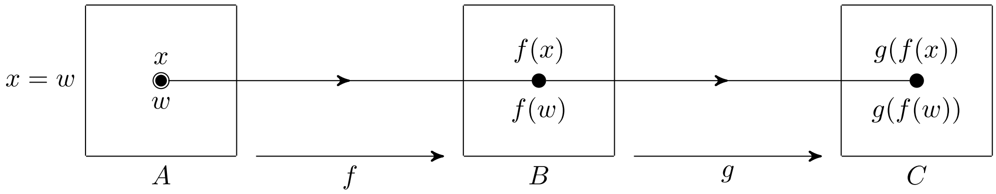
(2) Then $g(f(x)) = g(f(w))$.



(3) [Note that $f(x), f(w) \in B$.] By the injectivity of g, since $g(f(x)) = g(f(w))$, we have $f(x) = f(w)$.



(4) [Note that $x, w \in A$.] By the injectivity of f, since $f(x) = f(w)$, we have $x = w$.



It follows that $g \circ f$ is injective.

Proof of Statement (2) of Theorem ($\sharp_1$) (without pictures).

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose f, g are injective. [We want to verify that $g \circ f$ is injective under this assumption. By the definition of injectivity, this is the same as verify that for any $x, w \in A$, if $(g \circ f)(x) = (g \circ f)(w)$ then $x = w$.]

Pick any $x, w \in A$. [We want to argue that for the same x, w, if $(g \circ f)(x) = (g \circ f(w))$ then $x = w$.]
Suppose $(g \circ f)(x) = (g \circ f)(w)$. Then $g(f(x)) = g(f(w))$.
By the injectivity of g, since $g(f(x)) = g(f(w))$, we have $f(x) = f(w)$.
By the injectivity of f, since $f(x) = f(w)$, we have $x = w$.

It follows that $g \circ f$ is injective.

Very formal proof of Statement (2) of Theorem ($\sharp_1$).

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose f, g are injective. [We want to verify that under the above assumption, $g \circ f$ is injective.]

Pick any objects x, w.

I. Suppose $x \in A$ and $w \in A$. [Assumption.]

II. Suppose $(g \circ f)(x) = (g \circ f)(w)$. [Assumption.]

III. $f(x) \in B$ and $f(w) \in B$. [**I**, definition of function.]

IV. $g(f(x)) = g(f(w))$ [**II**.]

V. g is injective. [Assumption.]

VI. $f(x) = f(w)$. [**III**, **IV**, **V**, definition of injectivity.]

VII. f is injective. [Assumption.]

VIII. $x = w$. [**I**, **VI**, **VII**, definition of injectivity.]

Hence $g \circ f$ is injective.

5. **Theorem** ($\sharp_2$).

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. The following statements hold:

(1) *Suppose $g \circ f$ is surjective. Then g is surjective.*

(2) *Suppose $g \circ f$ is injective. Then f is injective.*

Proof of Theorem ($\sharp_2$). Exercise.

Remark.

The statements below are false. Dis-prove each of them by giving a counter-example.

(1) *Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is surjective. Then f is surjective.*

(2) *Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is injective. Then g is injective.*

Further remark.

Which of the statements below are true? Which are false?

(1) *Let A, B be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ be functions. Suppose $g \circ f$ is surjective. Then $f \circ g$ is surjective.*

(2) *Let A, B be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ be functions. Suppose $g \circ f$ is injective. Then $f \circ g$ is injective.*

(3) *Let A be a set, and $f, g : A \longrightarrow A$ be functions. Suppose $g \circ f$ is surjective. Then $f \circ g$ is surjective.*

(4) *Let A be a set, and $f, g : A \longrightarrow A$ be functions. Suppose $g \circ f$ is injective. Then $f \circ g$ is injective.*

They are all false. (Counter-examples?)