MATH1050 Intermediate Value Theorem, and the surjectivity and injectivity for continuous real-valued functions of one real-variable.

1. **Intermediate-Value Theorem. (Bolzano's version.)**

   Let $a, b \in \mathbb{R}$, with $a < b$. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose $f(a)f(b) < 0$. Suppose f is continuous on $[a, b]$. Then there exists some $x_0 \in (a, b)$ such that $f(x_0) = 0$.

   Remarks.

   (a) **Interpretation in terms of solving equations.**
   For such a function f, the equation $f(u) = 0$ with real unknown u has at least one solution in the interval (a, b).

   (b) **Pictorial interpretation.**
   The curve $y = f(x)$ cuts the x-axis at some point with x-coordinate in the interval (a, b).

   Intermediate-Value Theorem. ('General' version.)

   Let $a, b \in \mathbb{R}$, with $a < b$. Let $h : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose $h(a) \neq h(b)$. Suppose h is continuous on $[a, b]$. Then, for any $\gamma \in \mathbb{R}$, if γ is strictly between $h(a)$ and $h(b)$ then there exists some $c \in (a, b)$ such that $h(c) = \gamma$.

   Remarks.

   (a) **Interpretation in terms of solving equations.**
   For such a function h, for whichever real number γ strictly between $h(a)$ and $h(b)$, the equation $h(u) = \gamma$ with real unknown u has at least one solution in the interval (a, b).

   (b) **Pictorial interpretation.**
   For such a function h, for whichever real number γ strictly between $h(a)$ and $h(b)$, the horizontal line $y = \gamma$ intersects the curve $y = h(x)$ at some point with x-coordinate inside the interval (a, b).

   The two versions of the Intermediate Value Theorem are logically equivalent to each other. The justification of the logical equivalence is easy and is left as an exercise.

   The proof of the Intermediate Value Theorem will be given in your *Analysis* course.

2. **Definition. (Strict monotonicity.)**

   Let I be an interval, and $h : D \longrightarrow \mathbb{R}$ be a real-valued function of one real variable with domain D which contains I as a subset entirely.

   (a) *h is said to be* **strictly increasing** *on I if the statement* (St-I) *holds:*
   (St-I) *For any $s, t \in I$, if $s < t$ then $h(s) < h(t)$.*

   (b) *h is said to be* **strictly decreasing** *on I if the statement* (St-D) *holds:*
   (St-D) *For any $s, t \in I$, if $s < t$ then $h(s) > h(t)$.*

   (c) *h is said to be* **strictly monotonic** *on I if h is strictly increasing on I or h is strictly decreasing on I.*

   Remark. When $D = I$, we may simply omit the words '*on I*'

3. **Theorem (1).**

   Let J be an interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.
   Suppose g is strictly monotonic on J. Then g is injective.

   Remark. The proof of Theorem (1) is easy and is left as an exercise.

   Further remark. The converse of Theorem (1), which is the statement (†) below, is false:

   (†) *Let J be an interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.*
   Suppose g is injective. Then g is strictly monotonic on J.

   One counter-example against the statement (†) is provided by the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

   $$g(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ -x & \text{if} \quad x \in \mathbb{R}\backslash\mathbb{Q} \end{cases} .$$

   In fact, although g is injective, g fails to be strictly monotonic on any interval at all. It seems that the problem for the function g is that its graph is 'broken' (or more formally, g is not continuous).

4. **Theorem (2).**

   Let J be an interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.

   Suppose g is continuous on J.

   Suppose g is injective. Then g is strictly monotonic on J.

   Remark. The proof of Theorem (2) relies on Lemma (3).

   Further remark. It is impossible for a real-valued function of one real-variable, having an interval as a domain and being continuous throughout that interval, to be injective but to fail to be strictly monontonic on that interval.

   Illustration. Refer to the Handout *Surjectivity and injectivity for 'nice' real-valued functions of one real variable.* There we verify that the functions $f_1, f_2, f_5, f_6 : \mathbb{R} \longrightarrow \mathbb{R}$ given by

   $$f_1(x) = 0.1x^3, \quad f_2(x) = \sqrt[5]{x} - 1, \quad f_5(x) = 1.3^x, \quad f_6(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \text{for any } x \in \mathbb{R}$$

   are injective. It seemst that coincidentally they are strictly monotonic on $\mathbb{R}$.

   This is no coincidence at all, in light of the fact that each of them is continuous on $\mathbb{R}$. Theorem (2) guarantees that each of them is strictly monotonic because it is continuous on $\mathbb{R}$ and it is injective.

5. **Lemma (3).**

   Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \longrightarrow \mathbb{R}$ be a function.

   Suppose f is continuous on $[a, b]$.

   Suppose f is injective, and $\left\{ \begin{array}{c} f(a) < f(b) \\ f(a) > f(b) \end{array} \right\}$. *Then f is* $\left\{ \begin{array}{c} \textit{strictly increasing} \\ \textit{strictly decreasing} \end{array} \right\}$ *on $[a, b]$.*

   Remark. Lemma (3) is a consequence of the Intermediate Value Theorem.

6. **Proof of Lemma (3).**

   Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \longrightarrow \mathbb{R}$ be a function.

   Suppose f is continuous on $[a, b]$.

   Suppose f is injective.

   [We want to verify that if $f(a) < f(b)$ then f is strictly increasing on $[a, b]$.
   The argument for the statement '*if $f(a) > f(b)$ then f is strictly decreasing on* $[a, b]$' is similar.]

   Suppose $f(a) < f(b)$.

   Pick any $s, t \in [a, b]$. Suppose $s < t$. Then by injectivity, $f(s) \neq f(t)$. Therefore $f(s) < f(t)$ or $f(s) > f(t)$. [We want to rule out '$f(s) > f(t)$'. Ask: What is wrong with it?]

   We verify that it is impossible for '$f(s) > f(t)$' to hold:

   Suppose it were true that $f(s) > f(t)$.

   - (Case 1.) Suppose $f(a) \geq f(s)$. Then $f(b) > f(a) \geq f(s) > f(t)$. Therefore $t < b$.
     Note that f is continuous on $[t, b]$.
     Therefore by the Intermediate Value Theorem, there would exist some $u \in (t, b)$ such that $f(u) = f(a)$.
     But $u > a$. Contradiction arises.
   - (Case 2.) Suppose $f(a) < f(s)$. Then $a < s < t$. Therefore, by injectivity, $f(a) \neq f(t)$.
     * (Case 2a.) Suppose $f(a) < f(t)$.
       Note that f is continuous on $[a, s]$.
       Then by the Intermediate Value Theorem, there would exist some $v \in (a, s)$ such that $f(v) = f(t)$.
       But $v < t$. Contradiction arises.
     * (Case 2b.) Suppose $f(a) > f(t)$.
       Note that f is continuous on $[s, t]$.
       Then by the Intermediate Value Theorem, there would exist some $w \in (s, t)$ such that $f(w) = f(a)$.
       But $a < w$. Contradiction arises.

   Therefore in any case, contradiction arises.

   Hence $f(s) < f(t)$ in the first place.

   It follows that in the first place, f is strictly increasing on $[a, b]$.

7. **Proof of Theorem (2).**

Let J be an interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.

Suppose g is continuous on J.

Suppose g is injective.

Further suppose g were were not strictly monotonic on J.

[We are going to look for a contradiction.]

By assumption, g would be neither strictly increasing on J nor strictly decreasing on J. [Ask. So what may go wrong?]

Since g was not strictly increasing on J, there would exist some $p, q \in J$ such that $p < q$ and $g(p) > g(q)$. Then, by Lemma (3), g would be strictly decreasing on $[p, q]$.

Since g was not strictly decreasing on J, there would exist some $s, t \in J$ such that $s < t$ and $g(s) < g(t)$. Then, by Lemma (3), g would be strictly increasing on $[s, t]$.

Since g was strictly decreasing on $[p, q]$ and strictly increasing on $[s, t]$, we would have $(p, q) \cap (s, t) = \emptyset$. (Why?)

Without loss of generality, suppose $q \leq s$. Then $p < q \leq s < t$. By injectivity, $g(p) \neq g(t)$.

- (Case 1.) Suppose $g(p) < g(t)$. Then by Lemma (3), g would be strictly increasing on $[p, t]$. Therefore $g(p) < g(q)$. Recall that $g(p) > g(q)$ also. Contradiction arises.
- (Case 2.) Suppose $g(p) > g(t)$. Then by Lemma (3), g would be strictly decreasing on $[p, t]$. Therefore $g(s) > g(t)$. Recall that $g(s) < g(t)$ also. Contradiction arises.

Hence, in any case, contradiction arises.

It follows that in the first place, g is strictly increasing on J or g is strictly decreasing on J.

8. **Definition. (Boundedness for real-valued functions of one real variable.)**

Let I be an interval, and $h : D \longrightarrow \mathbb{R}$ be a real-valued function of one real variable with domain D which contains I as a subset entirely.

(a) *Let $\beta \in \mathbb{R}$.*

h is said to be $\left\{ \begin{array}{c} \textbf{bounded above} \\ \textbf{bounded below} \end{array} \right\}$ **in $\mathbb{R}$ by β on I** *if for any $x \in I$,* $\left\{ \begin{array}{c} h(x) \leq \beta \\ h(x) \geq \beta \end{array} \right\}$.

Such a number β is called a(n) $\left\{ \begin{array}{c} \textbf{upper bound} \\ \textbf{lower bound} \end{array} \right\}$ **of the function h in $\mathbb{R}$ on I.**

(b) *h is said to be* **bounded above in $\mathbb{R}$ on I** *if h has an upper bound in $\mathbb{R}$ on I.*

(c) *h is said to be* **bounded below in $\mathbb{R}$ on I** *if h has a lower bound in $\mathbb{R}$ on I.*

(d) *h is said to be* **bounded in $\mathbb{R}$ on I** *if h is bounded above in $\mathbb{R}$ on I and bounded below in $\mathbb{R}$ on I.*

Remark. When $D = I$, we simply omit the words '*on I*'.

9. **Theorem (4).**

Let I be an interval in $\mathbb{R}$, and $g : I \longrightarrow \mathbb{R}$ be a function.

Suppose g is surjective. Then g is neither bounded above in $\mathbb{R}$ nor bounded below in $\mathbb{R}$.

Proof of Theorem (4). Exercise. (This is easy.)

Remark. The converse of Theorem (4), which is the statement $(\dagger')$ below, is false:

$(\dagger')$ *Let I be an interval in $\mathbb{R}$, and $g : I \longrightarrow \mathbb{R}$ be a function.*
Suppose g is neither bounded above in $\mathbb{R}$ nor bounded below in $\mathbb{R}$. Then g is surjective.

One counter-example against the statement $(\dagger')$ is provided by the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$g(x) = \left\{ \begin{array}{lll} x & \text{if} & x \in \mathbb{Q} \\ 0 & \text{if} & x \in \mathbb{R} \backslash \mathbb{Q} \end{array} \right. .$$

In fact, although g is neither bounded above in $\mathbb{R}$ nor bounded below in $\mathbb{R}$, g fails to be surjective. It seems that the problem for the function g is that its graph is 'broken' (or more formally, g is not continuous).

10. **Theorem (5).**

Let J be an interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.

Suppose g is continuous on J. Further suppose g is neither bounded above in $\mathbb{R}$ nor bounded below in $\mathbb{R}$. Then g is surjective.

Remark. Theorem (5) is a consequence of the Intermediate Value Theorem.

Illustration. Refer to the Handout *Surjectivity and injectivity for 'nice' real-valued functions of one real variable.*

The function $f_3 : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f_3(x) = x^5 - 2x^3 + x$ for any $x \in \mathbb{R}$. We note that f_3 is continuous on $\mathbb{R}$. We further note that f_3 is neither bounded above in $\mathbb{R}$ nor bounded below in $\mathbb{R}$. (How?) In the light of this, we expect f_3 to be surjective according to Theorem (5).

Here we choose to verify the surjectivity of f_3 by directly applying the Intermediate-Value Theorem:

Pick any $y \in \mathbb{R}$. We have $y = 0$ or $y > 0$ or $y < 0$.

- $\star$ (Case 1). Suppose $y = 0$. Then $f_3(0) = 0 = y$.
- $\star$ (Case 2). Suppose $y > 0$. [We are going to name an appropriate positive number which is assigned by f_3 to some number strictly greater than y.]
  Note that $y + 2 > y + 1 > y > 0$, and $y + 3 > y + 1 > 1$.
  Then $f_3(y + 2) = (y + 2)[(y + 2)^2 - 1]^2 = (y + 1)^3(y + 3)^2 > y \cdot 1^2 \cdot 1^2 > y$.
  Also note that $f_3(0) = 0 < y$.
  By the Intermediate-Value Theorem, there exists some $x \in [0, y + 2]$ such that $y = f(x)$.
- $\star$ (Case 3). Suppose $y < 0$. Then $-y > 0$.
  By the argument in (Case 2), there exists some $w \in \mathbb{R}$ such that $f_3(w) = -y$. Take $v = -w$. We have $f_3(v) = f_3(-w) = -f_3(w) = -(-y) = y$.

It follows that f_3 is surjective.

It will turn out that the way we apply the Intermediate Value Theorem in this illustration is similar to how we apply the Intermediate Value Theorem in the proof for Theorem (5).

11. **Corollary (6).**

Let J be an open interval, with $J = (*, \star)$, and $g : J \longrightarrow \mathbb{R}$ be a function. (Here $*$ stands for a real number a or the symbol $-\infty$, and $\star$ stands for a real number b or the symbol $+\infty$.)

Suppose g is continuous on J.

Further suppose $\left\{ \begin{array}{c} \lim\limits_{x \to *^+} g(x) = -\infty \\ \lim\limits_{x \to *^+} g(x) = +\infty \end{array} \right\}$ *and* $\left\{ \begin{array}{c} \lim\limits_{x \to \star^-} g(x) = +\infty \\ \lim\limits_{x \to \star^-} g(x) = -\infty \end{array} \right\}$. *(Here $*^+$ stands for the symbol a^+ or the symbol $-\infty$, and $\star^-$ stands for the symbol b^- or the symbol $+\infty$.)*

Then g is surjective.

Remark. Corollary (6) is a consequence of Theorem (5).

12. **Proof of Theorem (5).**

Let J be an interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.

Suppose g is continuous on J. Further suppose g is neither bounded above in $\mathbb{R}$ nor bounded below in $\mathbb{R}$.

We verify that g is surjective:

Pick any $y \in \mathbb{R}$.

[Idea. We want to place y strictly between two numbers (with possibly 'large' magnitudes) which are 'outputs' of g. Then the Intermediate Value Theorem will guarantees that $y = g(u)$ with unknown u in J will have a solution.]

Since g is not bounded below in $\mathbb{R}$, there exists some $a \in J$ such that $g(a) < y$.

Since g is not bounded above in $\mathbb{R}$, there exists some $b \in J$ such that $g(b) > y$.

Note that $a \neq b$. Without loss of generality, assume $a < b$. Now note that $[a, b] \subset J$.

g is continuous on $[a, b]$. By definition, y is strictly between $g(a)$ and $g(b)$. Then by the Intermediate Value Theorem, there exists some $x \in (a, b)$ such that $g(x) = y$. By definition, $x \in J$.

It follows that g is surjective.