1. **Intermediate-Value Theorem. (Bolzano's version.)**

Let $a, b \in \mathbb{R}$, with $a < b$. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function.

Suppose $f(a)f(b) < 0$.

Suppose f is continuous on $[a, b]$.

Then there exists some $x_0 \in (a, b)$ such that $f(x_0) = 0$.

Intermediate-Value Theorem. ('General' version.)

Let $a, b \in \mathbb{R}$, with $a < b$. Let $h : [a, b] \longrightarrow \mathbb{R}$ be a function.

Suppose $h(a) \neq h(b)$.

Suppose h is continuous on $[a, b]$.

Then, for any $\gamma \in \mathbb{R}$, if γ is strictly between $h(a)$ and $h(b)$ then there exists some $c \in (a, b)$ such that $h(c) = \gamma$.

The two versions of the Intermediate Value Theorem are logically equivalent to each other. The justification of the logical equivalence is easy and is left as an exercise.

The proof of the Intermediate Value Theorem will be given in your *Analysis* course.

Intermediate Value Theorem (Bolzano's version)

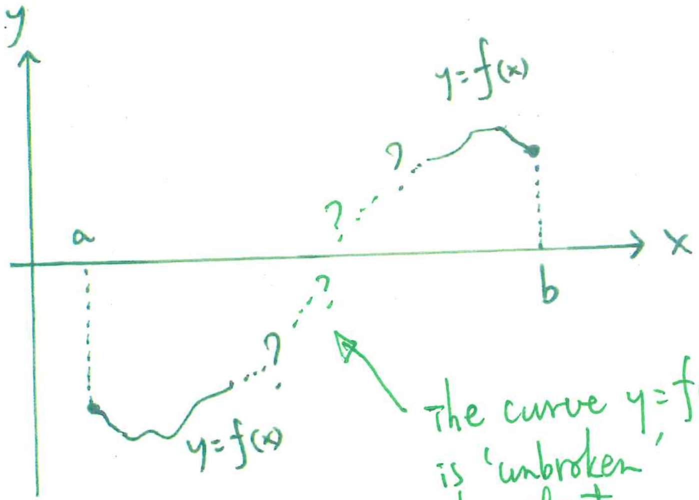
Assumptions:

- $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$.
- $f(a) \cdot f(b) < 0$.

Conclusion:

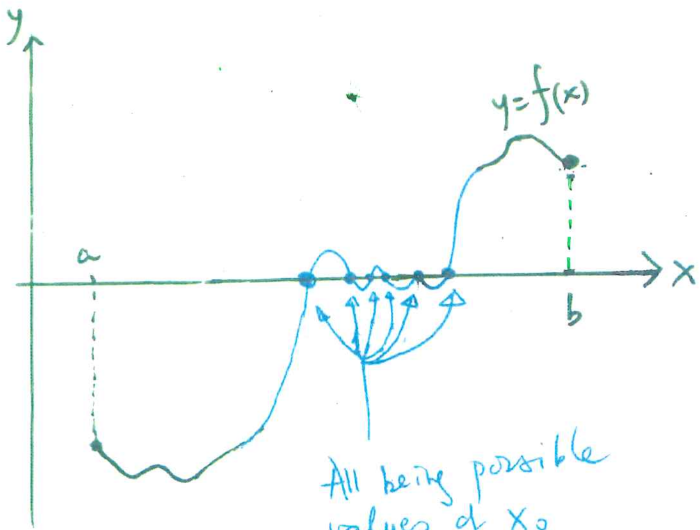
- There exists some $x_0 \in (a,b)$ such that $f(x_0) = 0$.

Picture for the scenario '$f(a) < 0 < f(b)$'.



The curve $y = f(x)$ is 'unbroken' throughout.

Similar story when '$f(a) < 0 < f(b)$' is replaced by '$f(a) > 0 > f(b)$'.



All being possible values of x_0

(In other words, the equation $f(u) = 0$ with unknown u has at least one solution in (a,b).)

Intermediate Value Theorem (General version)

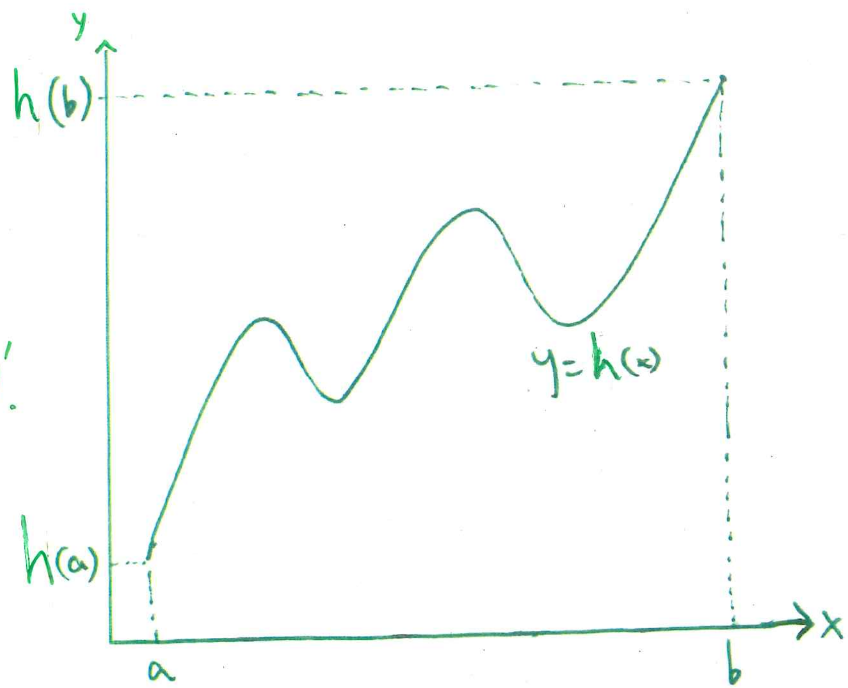
Assumptions:

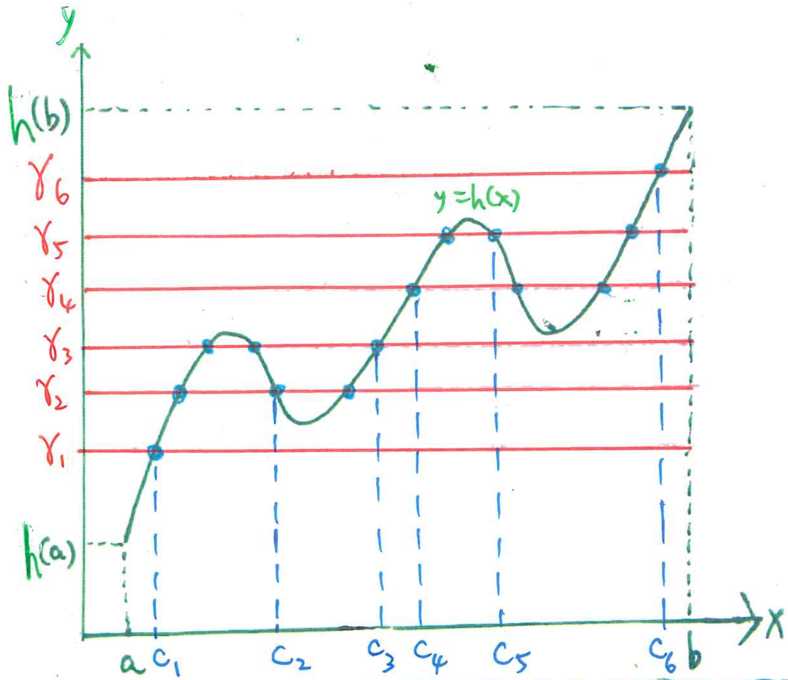
- $h: [a, b] \to \mathbb{R}$ is continuous on $[a, b]$.
- $h(a) \neq h(b)$.

Conclusion:

- For each γ strictly between $h(a)$ and $h(b)$, there exits some $c \in (a, b)$ such that $h(c) = \gamma$.

Picture for the scenario '$h(a) < h(b)$'.





- How to justify?

For each $\gamma \in (h(a), h(b))$, consider what can be said about the continuous function $h(x) - \gamma$ on $[a, b]$.

2. **Definition. (Strict monotonicity.)**

Let I be an interval, and $h : D \longrightarrow \mathbb{R}$ be a real-valued function of one real variable with domain D which contains I as a subset entirely.

(a) *h is said to be* **strictly increasing** *on I if the statement* (St-I) *holds:*

(St-I) *For any $s, t \in I$, if $s < t$ then $h(s) < h(t)$.*

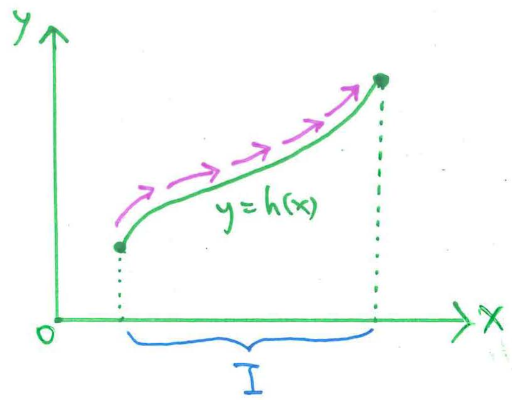
(b) *h is said to be* **strictly decreasing** *on I if the statement* (St-D) *holds:*

(St-D) *For any $s, t \in I$, if $s < t$ then $h(s) > h(t)$.*

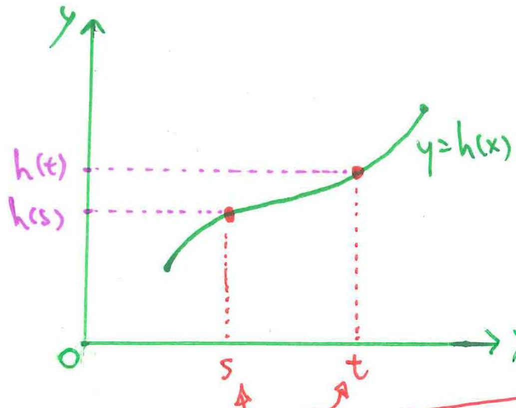
(c) *h is said to be* **strictly monotonic** *on I if h is strictly increasing on I or h is strictly decreasing on I.*

Remark. When $D = I$, we may simply omit the words '*on I*'

Picture for the notion 'strictly increasing function'.



Idea to be captured as:



Arbitrary $s, t \in I$ satisfying $s < t$: ask '$h(s) < h(t)$?'

⋮

Answer: 'yes'.

3. **Theorem (1).**

Let J be an interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.

Suppose g is strictly monotonic on J. Then g is injective.

Remark. The proof of Theorem (1) is easy and is left as an exercise.

Further remark. The converse of Theorem (1), which is the statement (†) below, is false:

(†) *Let J be an interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.*

Suppose g is injective. Then g is strictly monotonic on J.

One counter-example against the statement (†) is provided by the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \end{cases} .$$

In fact, although g is injective, g fails to be strictly monotonic on any interval at all. It seems that the problem for the function g is that its graph is 'broken' (or more formally, g is not continuous).

4. **Theorem (2).**

Let J be an interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.

Suppose g is continuous on J.

Suppose g is injective. Then g is strictly monotonic on J.

Remark. The proof of Theorem (2) relies on Lemma (3).

Further remark. It is impossible for a real-valued function of one real-variable, having an interval as a domain and being continuous throughout that interval, to be injective but to fail to be strictly monontonic on that interval.

5. **Lemma (3).**

Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \longrightarrow \mathbb{R}$ be a function.

Suppose f is continuous on $[a, b]$. Suppose f is injective, and $\left\{ \begin{array}{c} f(a) < f(b) \\ f(a) > f(b) \end{array} \right\}$.

Then f is $\left\{ \begin{array}{c} \textit{strictly increasing} \\ \textit{strictly decreasing} \end{array} \right\}$ *on $[a, b]$.*

Remark. Lemma (3) is a consequence of the Intermediate Value Theorem.

What? 'For any $s, t \in [a,b]$, if $s<t$ then $f(s)<f(t)$.'

6. **Proof of Lemma (3), for the case '$f(a) < f(b)$'.**

Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \longrightarrow \mathbb{R}$ be a function.

Suppose f is continuous on $[a, b]$. Suppose f is injective.

Suppose $f(a) < f(b)$. [Want to deduce: $\boxed{f \text{ is strictly increasing on } [a,b].}$]

Pick any $s, t \in [a, b]$. Suppose $s < t$. Then by injectivity, $f(s) \neq f(t)$.

Therefore $f(s) < f(t)$ or $f(s) > f(t)$. [We want to rule out '$f(s) > f(t)$'. Ask: What is wrong with it?]

- Suppose it were true that $f(s) > f(t)$.

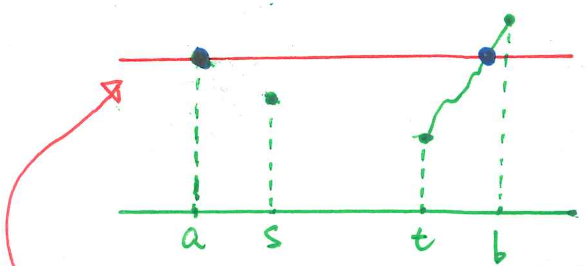
(Case 1.) Suppose $f(a) \geq f(s)$.

(Case 2.) Suppose $f(a) < f(s)$. (Note that $f(a) \neq f(t)$. How?)

(Case 2a.) Suppose $f(a) < f(t)$.

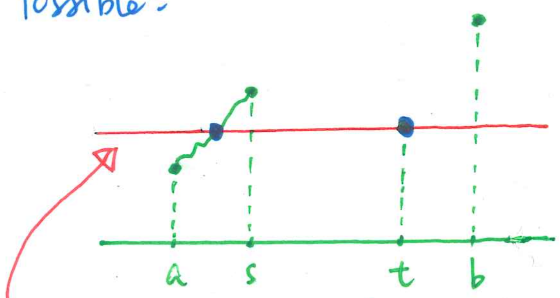
(Case 2b.) Suppose $f(a) > f(t)$.

Possible?



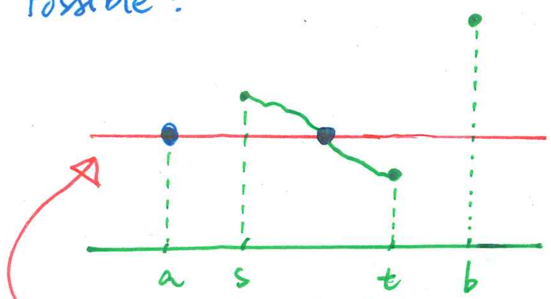
No, because of the Intermediate Value Theorem and injectivity of f.

Possible?



No, because of the Intermediate Value Theorem and injectivity of f.

Possible?



No, because of the Intermediate Value Theorem and injectivity of f.

In each case, contradiction arises.

It follows that $f(s) < f(t)$.

7. Proof of Theorem (2).

Let J be an interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.

Suppose g is continuous on J. Suppose g is injective.

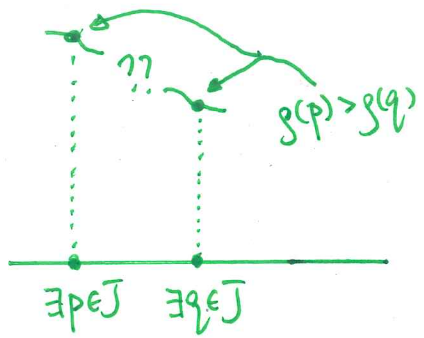
Further suppose g were were not strictly monotonic on J.

[We are going to look for a contradiction.]

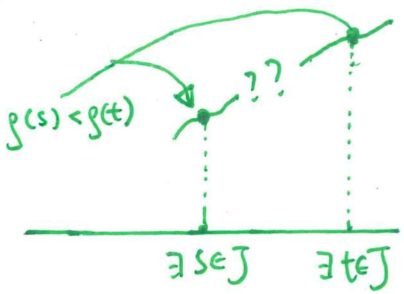
By assumption, g would be neither strictly increasing on J nor strictly decreasing on J.

[Ask. So what may go wrong?]

As g was not strictly increasing on J,



As g was not strictly decreasing on J,



Now, by Lemma (3),
g was strictly decreasing on $[p,q]$, and
g was strictly increasing on $[s,t]$.
Then $(p,q)\cap(s,t)=\emptyset$. (why?)

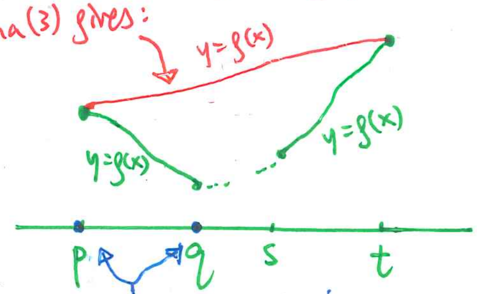
We now suppose, without loss of generality, that $q \leq s$.

Then $p<q\leq s<t$.

By injectivity, $g(p) \neq g(t)$.

(Case 1.) Suppose $g(p)<g(t)$.

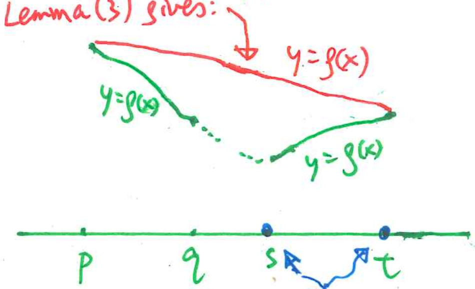
Lemma (3) gives:



Now $\begin{cases} g(p)>g(q) \\ g(p)<g(q) \end{cases}$ simultaneously.

(Case 2.) Suppose $g(p)>g(t)$.

Lemma (3) gives:



Now $\begin{cases} g(s)<g(t) \\ g(s)>g(t) \end{cases}$ simultaneously.

Contradiction arises in each case. Et cetera. □

8. **Definition. (Boundedness for real-valued functions of one real variable.)**

Let I be an interval, and $h : D \longrightarrow \mathbb{R}$ be a real-valued function of one real variable with domain D which contains I as a subset entirely.

(a) *Let $\beta \in \mathbb{R}$.*

h is said to be $\left\{ \begin{array}{c} \textbf{bounded above} \\ \textbf{bounded below} \end{array} \right\}$ **in $\mathbb{R}$ by β on** I *if for any $x \in I$,* $\left\{ \begin{array}{c} h(x) \leq \beta \\ h(x) \geq \beta \end{array} \right\}$.

Such a number β is called a(n) $\left\{ \begin{array}{c} \textbf{upper bound} \\ \textbf{lower bound} \end{array} \right\}$ **of the function h in $\mathbb{R}$ on** I.

(b) *h is said to be* **bounded above in $\mathbb{R}$ on** I *if h has an upper bound in $\mathbb{R}$ on I.*

(c) *h is said to be* **bounded below in $\mathbb{R}$ on** I *if h has a lower bound in $\mathbb{R}$ on I.*

(d) *h is said to be* **bounded in $\mathbb{R}$ on** I *if h is bounded above in $\mathbb{R}$ on I and bounded below in $\mathbb{R}$ on I.*

Remark. When $D = I$, we simply omit the words '*on I*'.

9. **Theorem (4).**

Let I be an interval in $\mathbb{R}$, and $g : I \longrightarrow \mathbb{R}$ be a function.

Suppose g is surjective. Then g is neither bounded above in $\mathbb{R}$ nor bounded below in $\mathbb{R}$.

Proof of Theorem (4). Exercise. (This is easy.)

Remark. The converse of Theorem (4), which is the statement $(\dagger')$ below, is false:

$(\dagger')$ *Let I be an interval in $\mathbb{R}$, and $g : I \longrightarrow \mathbb{R}$ be a function.*

Suppose g is neither bounded above in $\mathbb{R}$ nor bounded below in $\mathbb{R}$. Then g is surjective.

One counter-example against the statement $(\dagger')$ is provided by the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases} .$$

In fact, although g is neither bounded above in $\mathbb{R}$ nor bounded below in $\mathbb{R}$, g fails to be surjective. It seems that the problem for the function g is that its graph is 'broken' (or more formally, g is not continuous).

10. **Theorem (5).**

Let J be an interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.

Suppose g is continuous on J.

Further suppose g is neither bounded above in $\mathbb{R}$ nor bounded below in $\mathbb{R}$.

Then g is surjective.

Remark. Theorem (5) is a consequence of the Intermediate Value Theorem.

11. **Corollary (6).**

Let J be an open interval, with $J = (*, \star)$, and $g : J \longrightarrow \mathbb{R}$ be a function. (Here $*$ stands for a real number a or the symbol $-\infty$, and $\star$ stands for a real number b or the symbol $+\infty$.)

Suppose g is continuous on J.

Further suppose $\left\{ \begin{array}{c} \lim\limits_{x \to *^+} g(x) = -\infty \\ \lim\limits_{x \to *^+} g(x) = +\infty \end{array} \right\}$ *and* $\left\{ \begin{array}{c} \lim\limits_{x \to \star^-} g(x) = +\infty \\ \lim\limits_{x \to \star^-} g(x) = -\infty \end{array} \right\}$. *(Here $*^+$ stands for the symbol a^+ or the symbol $-\infty$, and $\star^-$ stands for the symbol b^- or the symbol $+\infty$.)*

Then g is surjective.

Remark. Corollary (6) is a consequence of Theorem (5).

12. **Proof of Theorem (5).**

Let J be an interval in $\mathbb{R}$, and $g : J \longrightarrow \mathbb{R}$ be a function.

Suppose g is continuous on J. Further suppose g is neither bounded above in $\mathbb{R}$ nor bounded below in $\mathbb{R}$.

We verify that g is surjective: [Want to verify: For any $y \in \mathbb{R}$, there exists some $x \in J$ such that $y = g(x)$.]

Pick any $y \in \mathbb{R}$. [Try to name some $x \in J$ for which $y = g(x)$.]

[Idea. We want to place y strictly between two numbers (with possibly 'large' magnitudes) which are 'outputs' of g. Then the Intermediate Value Theorem will guarantees that $y = g(u)$ with unknown u in J will have a solution.]

Since g is not bounded below in $\mathbb{R}$, y is not a lower bound of g in $\mathbb{R}$ on J.
Then there exists some $a \in J$ such that $g(a) < y$.
Since g is not bounded above in $\mathbb{R}$, y is not an upper bound of g in $\mathbb{R}$ on J.
Then there exists some $b \in J$ such that $g(b) > y$.
Since $g(a) < y < g(b)$, we have $a \neq b$.
Without loss of generality, assume $a < b$. Since J is an interval, we have $[a, b] \subset J$.
Now:
g is continuous on $[a, b]$, and $g(a) < y < g(b)$.
By the Intermediate Value Theorem, there exists some $x \in (a, b)$ such that $y = g(x)$. By definition $x \in J$.
□