MATH1050 Surjectivity and injectivity for 'nice' real-valued functions of one real variable.

1. Let $f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8 : \mathbb{R} \longrightarrow \mathbb{R}$ be the functions defined by

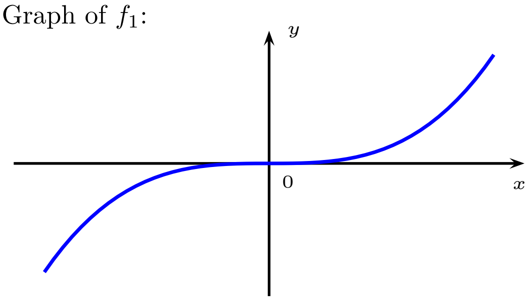
$$f_1(x) = 0.1x^3, \qquad f_2(x) = \sqrt[5]{x} - 1, \qquad f_3(x) = x^5 - 2x^3 + x, \qquad f_4(x) = 0.25x^2 \sin(10x),$$

$$f_5(x) = 1.3^x, \qquad f_6(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad f_7(x) = \frac{1}{x^2+1}, \qquad f_8(x) = 4^{\sin(4x)}$$

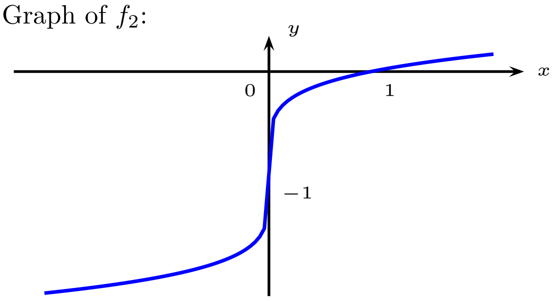
for any $x \in \mathbb{R}$.

Here are rough sketches of the respective graphs of the above functions:

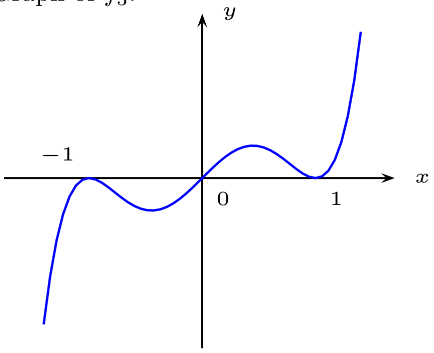
Graph of f_1:



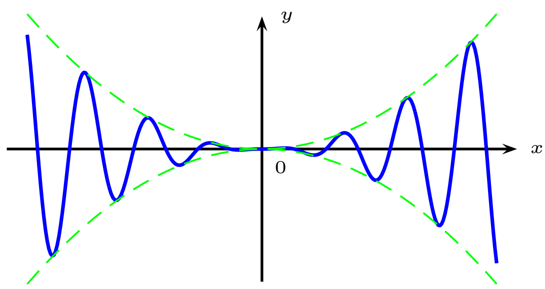
Graph of f_2:



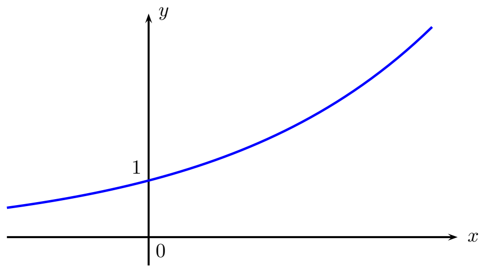
Graph of f_3:



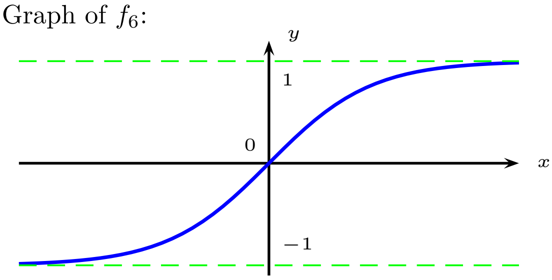
Graph of f_4:



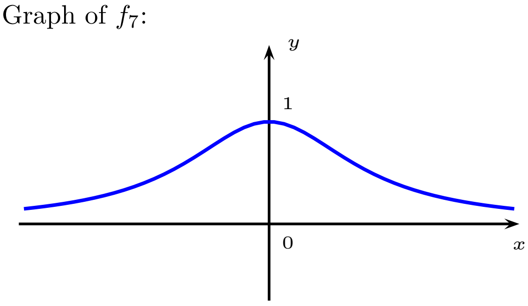
Graph of f_5:



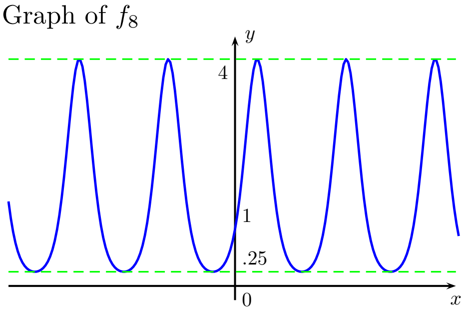
Graph of f_6:



Graph of f_7:



Graph of f_8



2. Which of $f_1, \cdots, f_8$ is/are surjective? Which not?

- f_1, f_2, f_3, f_4 are surjective.
- f_5, f_6, f_7, f_8 are not surjective.

Question. How to see which is surjective and which not, for such functions from $\mathbb{R}$ to $\mathbb{R}$?

Answer (b1). Inspect the graph of $f_1, \cdots, f_8$ on the 'coordinate plane'.

- $i = 1, 2, 3, 4$. Why surjective?
  At each 'altitude' $b \in \mathbb{R}$, the horizontal line $y = b$ cuts the graph of f_i at least once. Some $x_b \in \mathbb{R}$ satisfies $y = f_i(x_b)$.
- $i = 5, 6, 7, 8$. Why not surjective?
  At some 'altitude' $b_0 \in \mathbb{R}$, the horizontal line $y = b_0$ cuts the graph of f_i nowhere. No $x \in \mathbb{R}$ satisfies $b_0 = f_i(x)$.

Answer (b2). We re-interpret (b1) in terms of solving equations.

- $i = 1, 2, 3, 4$. Why surjective? For each $b \in \mathbb{R}$, the equation $b = f_i(u)$ with 'unknown' u in $\mathbb{R}$ has at least one solution in $\mathbb{R}$.
- $i = 5, 6, 7, 8$. Why not surjective? There is some $b_0 \in \mathbb{R}$ for which the equation $b_0 = f(u)$ with 'unknown' u in $\mathbb{R}$ has no solution in $\mathbb{R}$.

Answer (a). Directly verify the condition (S) or its negation respectively.

- $i = 1, 2, 3, 4$. [Recall the statement (S).]
  - How do we check the surjectivity of f_1, in practice?
    Pick any $y \in \mathbb{R}$. [This y is kept fixed in the discussion below.]
    [We name a candidate $x \in \mathbb{R}$ for which $y = f_1(x)$. An appropriate candidate is given by a solution of the equation $y = f_1(u)$ with unknown u in $\mathbb{R}$.]
    Take $x = \sqrt[3]{10y}$. By definition, $x \in \mathbb{R}$.
    We have $f_1(x) = 0.1x^3 = 0.1(\sqrt[3]{10y})^3 = 0.1(10y) = y$.
  - How about f_2? [Exercise.]

  Remark. Things are more difficult in practice for f_3, f_4, when we do not make use of the calculus. (Why?)
- $i = 5, 6, 7, 8$. [Recall the statement $\sim(S)$.]
  - How do we check the non-surjectivity of f_8, in practice?
    [Name a candidate $y_0 \in \mathbb{R}$ for which $y_0 \neq f_8(x)$ for any $x \in \mathbb{R}$. We are aware that for any $x \in \mathbb{R}$, $4^{-1} \leq 4^{\sin(4x)} \leq 4$.]
    Take $y_0 = 5$. Pick any $x \in \mathbb{R}$. We have $f_8(x) = 4^{\sin(4x)} \leq 4 < 5$. Hence $f_8(x) \neq y_0$.
  - How about f_6?
    [How to find a candidate y_0 satisfying $y_0 \neq f(x)$ for any $x \in \mathbb{R}$? Look for a necessary condition for the statement '$x, y \in \mathbb{R}$ and $y = f_6(x)$'. For such x, y, we have $|y| = \left|\frac{e^x - e^{-x}}{e^x + e^{-x}}\right| = \frac{|e^x - e^{-x}|}{e^x + e^{-x}} \leq \frac{|e^x| + |e^{-x}|}{e^x + e^{-x}} = 1$. Now a candidate y_0 can be chosen in $\mathbb{R} \backslash [-1, 1]$.]
    Take $y_0 = 2$. For any $x \in \mathbb{R}$, we have $|f_6(x)| = \left|\frac{e^x - e^{-x}}{e^x + e^{-x}}\right| \leq 1 < 2$. Then $f_6(x) \neq y_0$.
  - How about f_5, f_7? [Exercise.]

3. Which of $f_1, \cdots, f_8$ is/are injective? Which not?

- f_1, f_2, f_5, f_6 are injective.
- f_3, f_4, f_7, f_8 are not injective.

Question. How to see which is injective and which not, for such functions from $\mathbb{R}$ to $\mathbb{R}$?

Answer (b1). Inspect the graph of $f_1, \cdots, f_8$ on the 'coordinate plane'.

- $i = 1, 2, 5, 6$. Why injective?
  At each 'altitude' $b \in \mathbb{R}$, the horizontal line $y = b$ cuts the graph of $f_i$at most once: no two distinct x, w satisfy $f_i(x) = f_i(w)$.
- $i = 3, 4, 7, 8$. Why not injective?
  At some 'altitude' $b_0 \in \mathbb{R}$, the horizontal line $y = b_0$ cuts the graph of f_i twice or more: some distinct x, w satisfy $f_i(x) = f_i(w)$.

Answer (b2). We re-interpret (b1) in terms of solving equations.

- $i = 1, 2, 5, 6$. Why injective?
  For each $b \in \mathbb{R}$, the equation $b = f_i(u)$ with 'unknown' u in $\mathbb{R}$ has at most one solution in $\mathbb{R}$.
- $i = 3, 4, 7, 8$. Why not injective? There is some value $b_0 \in \mathbb{R}$ for which the equation $b_0 = f_i(u)$ with 'unknown' u in $\mathbb{R}$ has two or more solutions in $\mathbb{R}$.

Answer (a). Directly verify the condition (I) or its negation respectively.

- $i = 1, 2, 5, 6$. [Recall the statement (I).]
  * How do we check the injectivity of f_6, in practice?
    Pick any $x, w \in \mathbb{R}$. [These x, w are fixed in the discussion below. We verify that if $f_6(x) = f_6(w)$ then $x = w$.]
    Suppose $f_6(x) = f_6(w)$.
    Then $\dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^w - e^{-w}}{e^w + e^{-w}}$.
    Therefore $e^{x+w} + e^{x-w} - e^{-x+w} - e^{-x-w} = (e^x - e^{-x})(e^w + e^{-w}) = (e^x + e^{-x})(e^w - e^{-w}) = e^{x+w} - e^{x-w} + e^{-x+w} - e^{-x-w}$.
    Hence $2e^{x-w} = 2e^{-x+w}$. We have $x - w = -x + w$. Then $x = w$.
  * How about f_1?
    Pick any $x, w \in \mathbb{R}$. Suppose $f_1(x) = f_1(w)$.
    Then $0.1x^3 = f_1(x) = f_1(w) = 0.1w^3$.
    We have $0 = x^3 - w^3 = (x - w)(x^2 + xw + w^2)$.
    Then $x - w = 0$ or $x^2 + xw + w^2 = 0$.
    (Case 1). Suppose $x - w = 0$. Then $x = w$.
    (Case 2). Suppose $x^2 + xw + w^2 = 0$. Then $\dfrac{x^2}{2} + \dfrac{w^2}{2} + \dfrac{(x+w)^2}{2} = 0$. So $x = w = 0$.
    In any case $x = w$.
  * How about f_2, f_5? [Exercise.]
- $i = 3, 4, 7, 8$. [Recall the statement $\sim(I)$.]
  * How do we check the non-injectivity of f_7, in practice?
    [Name $x_0, w_0 \in \mathbb{R}$ for which $x_0 \neq w_0$ and $f_7(x_0) = f_7(w_0)$. Try this roughwork: Start with the 'relation' '$f_7(x_0) = f_7(w_0)$' to see what may prevent us from obtaining '$x_0 = w_0$'.]
    Take $x_0 = \dfrac{1}{2}$, $w_0 = -\dfrac{1}{2}$. $x_0 \neq w_0$.
    $f_7(x_0) = \dfrac{1}{1 + (1/2)^2} = \dfrac{4}{5}$.
    $f_7(w_0) = \dfrac{1}{1 + (-1/2)^2} = \dfrac{4}{5}$. $f_7(x_0) = f_7(w_0)$.
  * How about f_3, f_4, f_8? [Exercise.]