MATH1050 Roots of polynomials with complex coefficients

1. **Definition. (Roots of polynomials.)**

   Let $u(x)$ be a polynomial with complex coefficients. Let α be a complex number.

   We say α is a **root of** $u(x)$ **in** $\mathbb{C}$ *if* $u(\alpha) = 0$.

   Remark. We observe that every polynomial with real coefficients is by definition a polynomial with complex coefficients.

2. **Theorem (1).**

   Suppose $u(x)$ is a non-constant polynomial with complex coefficients, with $\deg(u(x)) = n$.

   Then $u(x)$ has at most n distinct roots in $\mathbb{C}$.

   Proof of Theorem (1).

   Suppose $u(x)$ is a non-constant polynomial with complex coefficients, with $\deg(u(x)) = n$. By assumption $n \geq 1$.

   Suppose it were true that $u(x)$ had more than n distinct roots in $\mathbb{C}$. Suppose $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n$ were $n+1$ distinct roots of $u(x)$ in $\mathbb{C}$.

   By assumption, $u(\alpha_j) = 0$ for each $j = 0, 1, 2, \cdots, n$.

   In particular $u(\alpha_0) = 0$. By the Factor Theorem, there would exist some $u_1(x) \in \mathbb{C}[x]$ such that $u(x) = (x - \alpha_0)u_1(x)$ as polynomials.

   Note that $0 = u(\alpha_1) = (\alpha_1 - \alpha_0)u_1(\alpha_1)$. Since $\alpha_1 \neq \alpha_0$, we would have $u_1(\alpha_1) = 0$. By the Factor Theorem, there would exist some $u_2(x) \in \mathbb{C}[x]$ such that $u_1(x) = (x - \alpha_1)u_2(x)$ as polynomials.

   For the same $u_2(x) \in \mathbb{C}[x]$, we would have $u(x) = (x - \alpha_0)(x - \alpha_1)u_2(x)$ as polynomials.

   Repeating the above argument, we would deduce that there existed some $v(x) \in \mathbb{C}[x]$ such that $u(x) = (x - \alpha_0)(x - \alpha_1) \cdot ... \cdot (x - \alpha_n)v(x)$ as polynomials.

   By assumption $u(x)$ is not the zero polynomial. Then $v(x)$ is not the zero polynomial. (Why?) Therefore, we would have $n = \deg(u(x)) = n + 1 + \deg(v(x)) \geq n + 1 + 0 > n$. (Why?) Contradiction arises.

   Hence $u(x)$ has at most n distinct roots in $\mathbb{C}$ in the first place.

3. **Fundamental Theorem of Algebra.**

   Every non-constant polynomial with complex coefficients has a root in $\mathbb{C}$.

   Remark. The proof of the Fundamental Theorem of Algebra is beyond the scope of this course. In the discussion below, we take for granted the validity of the Fundamental Theorem of Algebra.

4. **Theorem (2). (Factorization of polynomials with complex coefficients into 'linear factors'.)**

   Suppose $u(x)$ is a non-constant polynomial with complex coefficients, with $\deg(u(x)) = n$, and with leading coefficient a_n.

   Then there exist some n complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$, not necessarily distinct, such that

   $$u(x) = a_n(x - \alpha_1)(x - \alpha_2) \cdot ... \cdot (x - \alpha_n) \qquad \text{as polynomials.}$$

   The n numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ are all the roots of $u(x)$ in $\mathbb{C}$.

   Proof of Theorem (2). Exercise. (Apply the Fundamental theorem of Algebra and the Factor Theorem to deduce that there exist some $\alpha \in \mathbb{C}$, and some degree-$(n-1)$ polynomial $v(x)$ with complex coefficients such that $u(x) = (x - \alpha)v(x)$ as polynomials. Next repeat this argument for $n - 1$ times. Now stop by virtue of Theorem (1).)

5. **Corollary (3). (Vieta's Theorem, relating roots and coefficients of polynomials.)**

Let $u(x)$ be a polynomial with complex coefficients, of degree $n \geq 1$, with its k-th coefficient being a_k for each $k = 0, 1, 2 \cdots, n$.

Suppose $\alpha_1, \alpha_2, \cdots, \alpha_n$ are all the n roots of $u(x)$ in $\mathbb{C}$.

Then

$$\begin{cases} \displaystyle\sum_{k=1}^{n} \alpha_k & = & -\dfrac{a_{n-1}}{a_n}, \\ \displaystyle\sum_{1 \leq j_1 < j_2 \leq n} \alpha_{j_1}\alpha_{j_2} & = & \dfrac{a_{n-2}}{a_n}, \\ \displaystyle\sum_{1 \leq j_1 < j_2 < j_3 \leq n} \alpha_{j_1}\alpha_{j_2}\alpha_{j_3} & = & -\dfrac{a_{n-3}}{a_n}, \\ & \vdots & \\ \alpha_1\alpha_2 \cdot \ldots \cdot \alpha_n & = & (-1)^n \cdot \dfrac{a_0}{a_n} \end{cases}$$

Proof of Corollary (3). This is a tedious exercise in 'comparing coefficients' for the two sides of the equality

$$a_0 + a_1 x + a_2(x) + \cdots + a_n x^n = a_n(x - \alpha_1)(x - \alpha_2) \cdot \ldots \cdot (x - \alpha_n) \quad \text{as polynomials.}$$

Remarks. Below are some special cases of Corollary (3), for 'polynomials of low degrees':

(a) Suppose $n = 2$. Then Corollary (3) gives $\alpha_1 + \alpha_2 = -\dfrac{a_1}{a_2}$, $\alpha_1\alpha_2 = \dfrac{a_0}{a_2}$.

These equalities relate the coefficients of the quadratic polynomial $u(x)$ with its sum of roots and its product of roots. You have learnt them in school maths.

(b) Suppose $n = 3$. Then Corollary (3) gives

$$\alpha_1 + \alpha_2 + \alpha_3 = -\frac{a_2}{a_3}, \quad \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = \frac{a_1}{a_3}, \quad \alpha_1\alpha_2\alpha_3 = -\frac{a_0}{a_3}.$$

(c) Suppose $n = 4$. Then Corollary (3) gives

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & = & -\dfrac{a_3}{a_4}, \\ \alpha_1\alpha_2 + \alpha_3\alpha_4 + \alpha_1\alpha_3 + \alpha_2\alpha_4 + \alpha_1\alpha_4 + \alpha_2\alpha_3 & = & \dfrac{a_2}{a_4}, \\ \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 & = & -\dfrac{a_1}{a_4} \\ \alpha_1\alpha_2\alpha_3\alpha_4 & = & \dfrac{a_0}{a_4}. \end{cases}$$

It is by playing with the expressions on the right-hand-sides of these equalities, known as 'symmetric polynomials of the α_j's', that mathematicians discovered the 'cubic formula' and the 'quartic formula' for solutions of cubic polynomial equations and quartic polynomial equations respectively, (analogous to the 'quadratic formula' of quadratic equations). These were all known by the end of the eighteenth century. A further investigation into the symmetry of such expressions for higher-degree polynomials led Galois and Abel onto the discovery that there is no formula as such for solutions of higher-degree polynomial equations. This will be the theme in the course *Fields and Galois Theory.*

6. **Theorem (4). ('Pairing-up' of complex roots for polynomials with real coefficients.)**

   Let $u(x)$ be a polynomial with real coefficients. Let α be a complex number.

   Suppose α is a root of $u(x)$ in $\mathbb{C}$.

   Then $\bar{\alpha}$ is also a root of $u(x)$ in $\mathbb{C}$.

   Proof of Theorem (4).

   Let $u(x)$ be a polynomial with real coefficients. For each $k \in \mathbb{N}$, denote by a_k the k-th coefficient of $u(x)$. Let α be a complex number. Suppose α is a root of $u(x)$ in $\mathbb{C}$. Then $0 = u(\alpha) = a_0 + a_1\alpha + a_2\alpha^2 + \cdots$.

   By assumption $a_k \in \mathbb{R}$ for each $k \in \mathbb{R}$.

   Then $0 = \overline{u(\alpha)} = \overline{a_0 + a_1\alpha + a_2\alpha^2 + \cdots} = \overline{a_0} + \overline{a_1\alpha} + \overline{a_2\alpha^2} + \cdots = a_0 + a_1\bar{\alpha} + a_2\bar{\alpha}^2 + \cdots = u(\bar{\alpha})$.

   Therefore $\bar{\alpha}$ is also a root of $u(x)$ in $\mathbb{C}$.

7. **Corollary (5).**

   Let $u(x)$ be a polynomial with real coefficients. Let α be a complex number.

   Suppose α is a non-real root of $u(x)$ in $\mathbb{C}$.

   Then $u(x)$ is divisible by the quadratic polynomial with real coefficients $x^2 - 2\mathsf{Re}(\alpha)x + |\alpha|^2$.

   Proof of Corollary (5). Exercise. (Apply the Factor Theorem.)

8. **Theorem (6). (Factorization of polynomials with real coefficients into linear or quadratic factors.)**

   Suppose $u(x)$ is a non-constant polynomial with real coefficients, with $\deg(u(x)) = n$.

   Then $u(x)$ factorizes as a product of linear polynomials with real coefficients and quadratic polynomials with real coefficients of negative discriminant.

   Proof of Theorem (6). Exercise. (Apply mathematical induction on the degrees of polynomials. At some stage the Division Algorithm for polynomials will be needed; see the handout *Basic results on polynomials 'beyond school mathematics'* for its statement.)

   Examples.

   (a) $x^3 - 1 = (x-1)(x^2 + x + 1)$ as polynomials.

   (b) $x^4 + 2x^2 + 1 = (x^2 + 1)^2$ as polynomials.

   (c) $x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$ as polynomials.

9. **Corollary (7).**

   Suppose $u(x)$ is a polynomial with real coefficients, of odd degree. Then $u(x)$ has at least one real root.

   Proof of Corollary (7). This is a consequence of Corollary (6).

   Remark. Below is a self-contained argument in the special case for cubic polynomials with real coefficients:

   - Let $u(x)$ be a degree-3 polynomial with real coefficients. By the Fundamental Theorem of Algebra, $u(x)$ has a root, say, α, in $\mathbb{C}$. If α is real, then $u(x)$ has a real root, namely α.

     Suppose α is non-real. Then $\bar{\alpha}$ is distinct from α, and is also a root of $u(x)$ in $\mathbb{C}$. By Corollary (5), $u(x)$ is divisible by the quadratic polynomial with real coefficients $x^2 - 2\mathsf{Re}(\alpha)x + |\alpha|^2$. There exists some $v(x) \in \mathbb{R}[x]$ such that $u(x) = (x^2 - 2\mathsf{Re}(\alpha)x + |\alpha^2|)v(x)$ as polynomials. It then follows that $v(x)$ is of degree 1. Hence there are some $a, b \in \mathbb{R}$ such that $a \neq 0$ and $v(x) = ax + b$ as polynomials. Now $-\frac{b}{a}$ is a real root of $u(x)$.