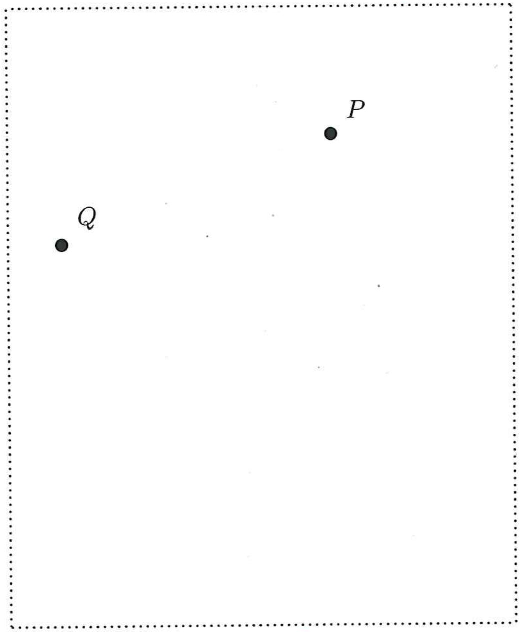
1. **Coordinate pairs and Cartesian plane in school mathematics.**

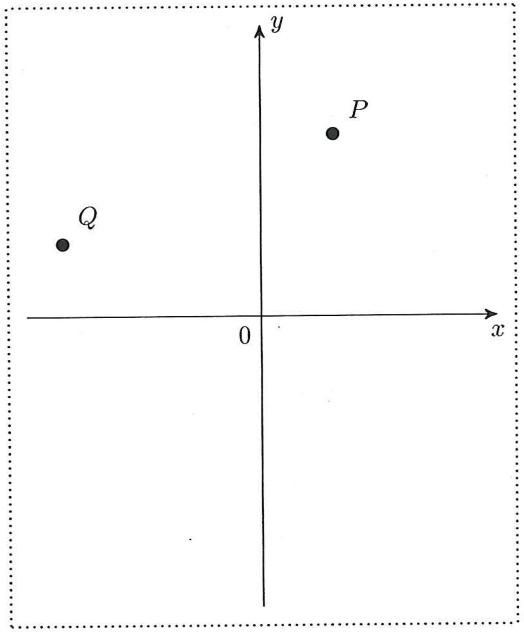
In school mathematics, we take the notions of coordinate pairs and the Cartesian (coordinate) plane for granted:

Euclidean plane (on which Euclidean geometry is true.)



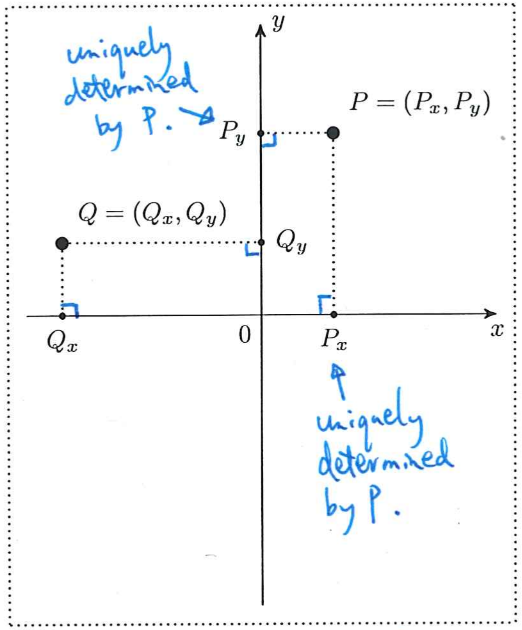
- 'Infinite in extent'.
- All points are in equal footing.
- All lines are in equal footing.

The mutually perpendicular lines 'x-axis', 'y-axis' are introduced.



- Each of 'x-axis', 'y-axis' is a copy of the 'real line' $\mathbb{R}$.

Cartesian coordinate plane.



- Each point is represented by exactly one 'ordered' pair of real numbers, uniquely determined by the point concerned.

This is then generalized to coordinate triples and the Cartesian (coordinate) space, and beyond.

Here we generalize the idea above in the context of set language.

2. **Ordered-ness in set language, and Cartesian product of two sets.**

Question. What is the essence in the notion of coordinate pairs in the plane?

- For any $s, t, u, v \in \mathbb{R}$, $((s,t) = (u,v)$ iff $(s = u$ and $t = v))$.

This guarantees that you will not confuse the point, say, (1, −1), with the point (−1, 1).

Imagine it makes sense to talk about the object called **ordered pair**

$$(s, t)$$

of

$$s, \qquad t$$

with s, t as first and second coordinate of (s, t).

But we require '$(\cdot, \cdot)$' to obey Convention ($\sharp$) below:

($\sharp$) *For any objects x, y, u, v, $((x,y) = (u,v)$ iff $(x = u$ and $y = v))$.*

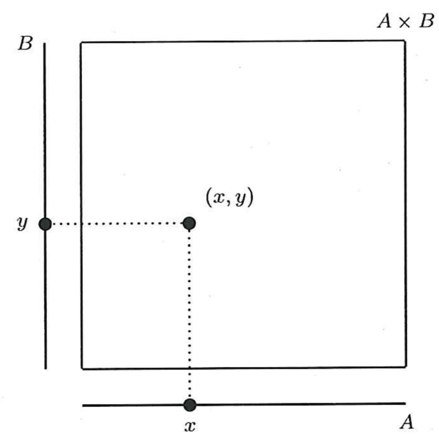
3. **Cartesian product of two sets.**

With this sense of ordered-ness in mind, it makes sense to define the notion of Cartesian product of two sets:

Definition.

Let A, B be sets. The **cartesian product** $A \times B$ *of the sets A, B is defined to be the set*

$$\{t \mid \text{ There exist some } x \in A, y \in B \text{ such that } t = (x, y)\}.$$



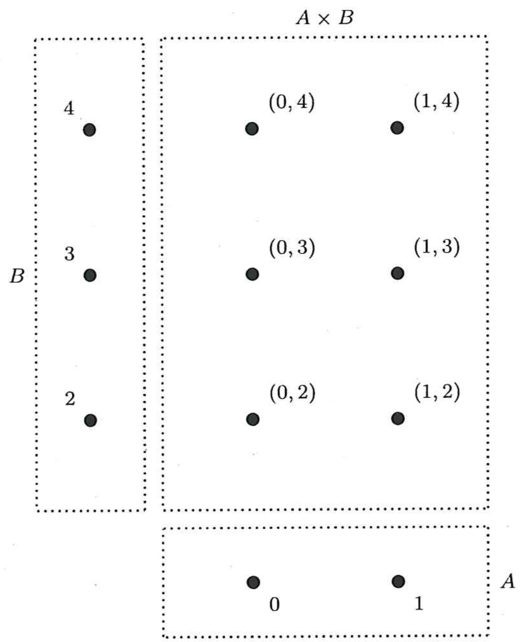
Remarks.

(1) 'Short-hand': $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$.

(2) When $A = B$, we write $A \times B$ as A^2.

Examples.

(a) $\{0,1\} \times \{2,3,4\} = \{(0,2),(0,3),(0,4),(1,2),(1,3),(1,4)\}$.



(b) $\mathbb{R} \times \mathbb{R}$ is the 'coordinate plane' $\mathbb{R}^2$ in school mathematics.

4. **Ordered pairs as set-theoretic objects.**

Philosphical question. *How to 'make sense' of the notion of ordered pairs in set language, in terms of objects already introduced in set language?*

Definition.

Let x, y be objects.

The **Kuratowski ordered pair** *of x, y, with x being the first coordinate and y being the second coordinate, is defined to be the set*

$$\{\{x\}, \{x, y\}\},$$

and is denoted by $(x, y)_K$.

> How to form this object from the objects x, y?
> ① Start with x, y.
> ② Form the sets $\{x\}, \{x, y\}$. Regard them as some objects.
> ③ Now form the set $\{\{x\}, \{x, y\}\}$.

This definition is appropriate because we have:

Lemma (OP).

Let x, y, u, v be objects. $(x, y)_K = (u, v)_K$ iff ($x = u$ and $y = v$).

Proof. Exercise in set language (playing around with '... $\in$ {...}' and logic).

Remark. From now on, we write $(x, y)_K$ as (x, y).

Further remark. Another version of definition for the notion of ordered pairs?

Wiener's version: $(x, y)_W = \{\{\emptyset, \{x\}\}, \{\{y\}\}\}$.

5. **Ordered triples and beyond.**

We define the notion for ordered triples in terms of ordered pairs.

Definition.

Let x, y, z be objects. We define the **ordered triple** *of x, y, z, with first, second, third coordinates being x, y, z respectively, to be $((x, y), z)$. We write it as (x, y, z).*

This definition is appropriate because we have:

Lemma (OT).

Let x, y, z, u, v, w be objects. $(x, y, z) = (u, v, w)$ iff ($x = u$ and $y = v$ and $z = w$).

Proof. Exercise.

Remark. We may extend the idea in the definition for the notion of ordered triple so as to give the definition for the notions of ordered quadruples, ordered quintuples et cetera.

6. **Theorem (∗). (Set-theoretic properties of cartesian products.)**

Let A, B, C, D be sets. The following statements hold:

(1) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

$(A \cap B) \times (C \cap D) = (A \times D) \cap (B \times C)$.

(2) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

$A \times (C \cup D) = (A \times C) \cup (A \times D)$.

(3) *Suppose* $A \subset B$ *and* $C \subset D$. *Then* $A \times C \subset B \times D$.

(4) *Suppose* $A \neq \emptyset$, $A \subset B$ *and* $A \times C \subset B \times D$. *Then* $C \subset D$.

(5) $A \times \emptyset = \emptyset$. $\quad \emptyset \times A = \emptyset$.