1. **Real-valued functions of one real variable in school mathematics**.

Below is a typical 'explanation' of the notion of real valued functions of one real variable in school mathematics:

Let D be a subset of $\mathbb{R}$ (very often $\mathbb{R}$ itself or $\mathbb{R}$ with a few points deleted).

A real-valued function defined on D is a 'rule of assignment' from D to $\mathbb{R}$, so that

each number in D is being assigned to exactly one element of $\mathbb{R}$.

When we refer to such a function by f, the set D will be referred to as the domain of this function f.

Whenever $x \in D$, $y \in \mathbb{R}$ and x is assigned to y, we write $y = f(x)$.

The set $G = \{(x, f(x)) \mid x \in D\}$ is called the graph of f. Note that $G \subset \mathbb{R}^2$.

How about 'general' functions?

~~Below is a typical 'explanation' of the notion of real valued functions of one real variable in school mathematics:~~

Let A, B be sets.

~~Let D be a subset of $\mathbb{R}$ (very often $\mathbb{R}$ itself or $\mathbb{R}$ with a few points deleted).~~

A ~~real valued~~ function ~~defined on D~~ from A to B is a 'rule of assignment' from ~~D~~ A to ~~$\mathbb{R}$~~ B, so that ~~each number in D~~ each element of A is being assigned to exactly one element of ~~$\mathbb{R}$~~ B.

When we refer to such a function by f, the set ~~D~~ A will be referred to as the domain of this function f. The set B is called the range of f.

Whenever $x \in$ ~~D~~ A, $y \in$ ~~$\mathbb{R}$~~ B and x is assigned to y, we write $y = f(x)$.

The set $G = \{(x, f(x)) \mid x \in D\}$ is called the graph of f. Note that $G \subset \mathbb{R}^2$.

??? A? ??? ???

2. **In-formal definition of function**.

Let A, B be sets.

A function from A to B is a 'rule of assignment' from A to B, so that

each element of A is being assigned to exactly one element of B.

Conventions and notations.

- When we denote such a function by f, we refer to it as $f : A \longrightarrow B$.
  Whenever $x \in A$, $y \in B$ and x is assigned to y, we write $y = f(x)$ (or $x \underset{f}{\mapsto} y$).
- A is called the **domain** of f. B is called the **range** of f.

Remark. We postpone the generalization of the notion of graphs of functions.

3. **'Blobs-and-arrows diagrams'.**

We may visualize a function by its '**blobs-and-arrows diagram**'.

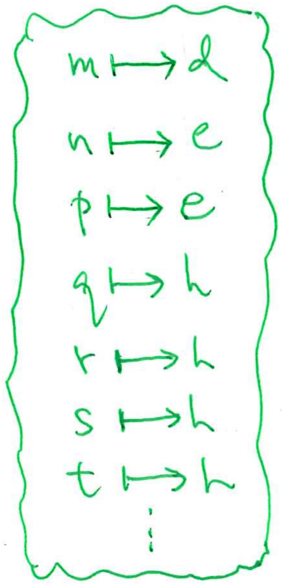
We illustrate the idea with the example below:

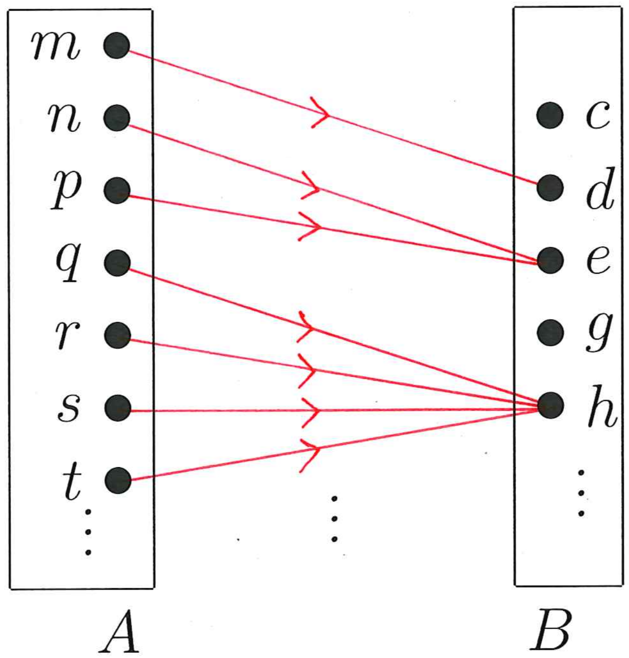
Let $A = \{m, n, p, q, r, s, t, ...\}$, $B = \{c, d, e, g, h, ...\}$, *and* $f : A \longrightarrow B$ *be defined by*

$$f(m) = d, f(n) = e, f(p) = e, f(q) = h, f(r) = h, f(s) = h, f(t) = h, \cdots .$$

By definition, f *assigns* m *to* d, n *to* e, p *to* e, q *to* h, r *to* h, s *to* h, t *to* h, $\cdots$.

We draw the 'blobs-and-arrows diagram' of the function f *as:*





4. **Notion of equality for functions.**

We regard two functions to be the same as each other exactly when they 'determine the same assignment'.

Definition.

Let A_1, A_2, B_1, B_2 be sets, and $f_1 : A_1 \longrightarrow B_1$, $f_2 : A_2 \longrightarrow B_2$ be functions.

We agree to say that f_1 is **equal** *to f_2 as functions, and to write $f_1 = f_2$, exactly when*

$$A_1 = A_2 \text{ and } B_1 = B_2 \text{ and } f_1(x) = f_2(x) \text{ for any } x \in A_1.$$

Examples and non-examples.

① $f_1 : \mathbb{R}\setminus\{1\} \to \mathbb{R}$ is given by

$f_1(x) = x+1$ for any $x \in \mathbb{R}\setminus\{1\}$;

$f_2 : \mathbb{R}\setminus\{1\} \to \mathbb{R}$ is given by

$f_2(x) = \dfrac{x^2-1}{x-1}$ for any $x \in \mathbb{R}\setminus\{1\}$.

- Are f_1, f_2 equal to each other?
- Yes.

② $g_1 : \mathbb{R} \to \mathbb{R}$ is given by

$g_1(x) = x+1$ for any $x \in \mathbb{R}$;

$g_2 : \mathbb{R}\setminus\{1\} \to \mathbb{R}$ is given by

$g_2(x) = \dfrac{x^2-1}{x-1}$ for any $x \in \mathbb{R}\setminus\{1\}$.

- Are g_1, g_2 equal to each other?
- No. (Reason: Domains do not agree.)

$g_3 : \mathbb{R} \to \mathbb{C}$ is given by

$g_3(x) = x+1$ for any $x \in \mathbb{R}$;

- Are g_1, g_3 equal to each other?
- No. (Reason: Ranges do not agree.)

③ $h_1 : \mathbb{R} \to \mathbb{R}$ is given by

$h_1(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$;

$h_2 : \mathbb{R} \to \mathbb{R}$ is given by

$h_2(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$.

- Are h_1, h_2 equal to each other?
- No. (Reason: 'Formulae' do not agree, for instance, at 0.)

5. **Compositions.**

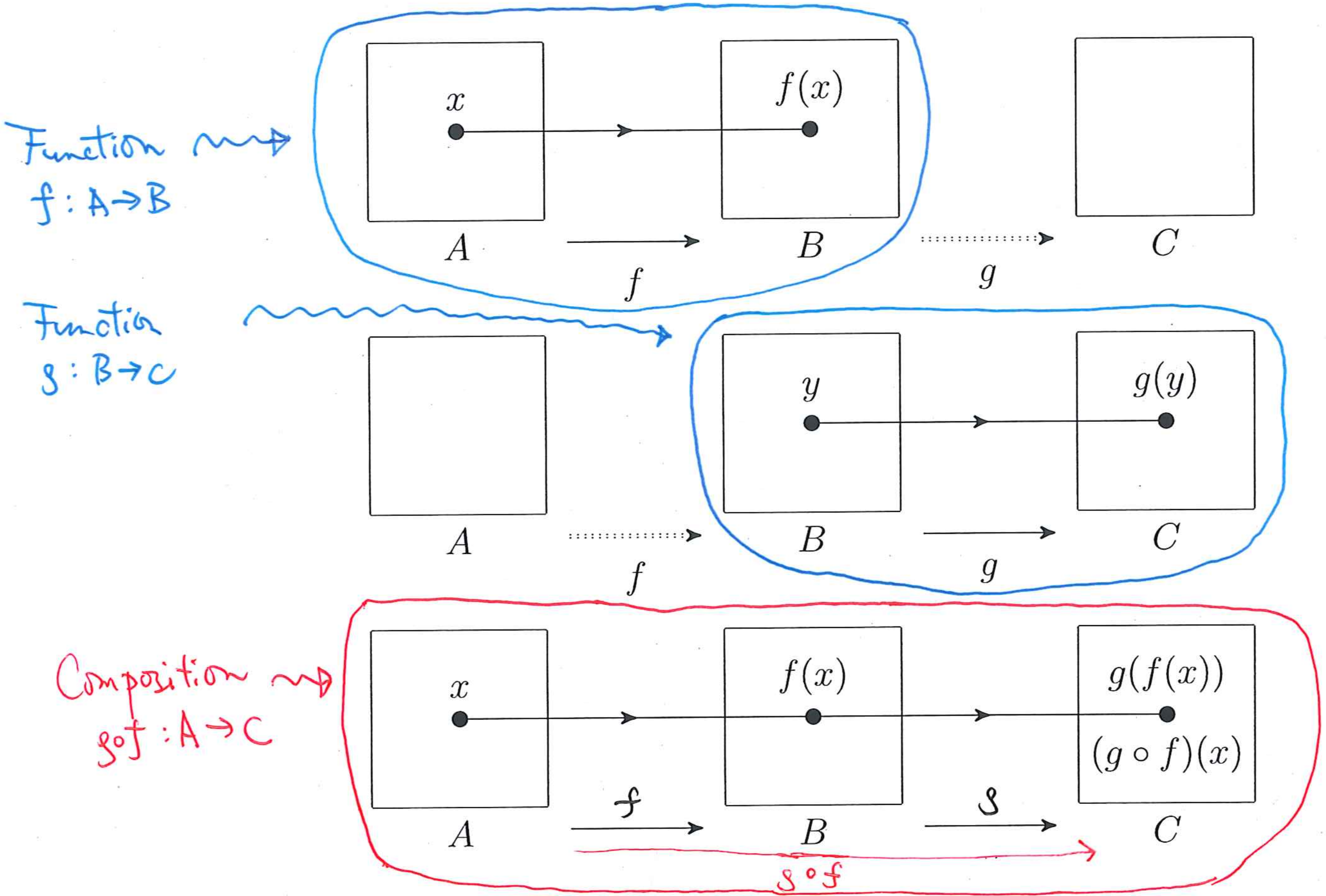
Out of two functions, the range of one being the same as the domain of the other, we may construct a third function

Definition.

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.

Define the function $g \circ f : A \longrightarrow C$ by $(g \circ f)(x) = g(f(x))$ for any $x \in A$.

$g \circ f$ is called the **composition** *of the functions f, g.*



Example of composition.
The function
'$x \longmapsto x^x$ for any $x \in (0, +\infty)$'
with domain $(0, +\infty)$ and range $\mathbb{R}$ is the composition $g \circ f$ in which:

- $f : (0, +\infty) \to \mathbb{R}$ is the function given by $f(x) = x\ln(x)$ for any $x \in (0, +\infty)$, and
- $g : \mathbb{R} \to \mathbb{R}$ is the function given by $g(y) = \exp(y)$ for any $y \in \mathbb{R}$.

Lemma (1). (Associativity of composition.)

Let A, B, C, D be sets, and

$$f : A \longrightarrow B, \quad g : B \longrightarrow C, \quad h : C \longrightarrow D$$

be functions.

$(h \circ g) \circ f = h \circ (g \circ f)$ as functions.

Remark. Hence there is no ambiguity when we refer to $(h \circ g) \circ f$ (and $h \circ (g \circ f)$) as $h \circ g \circ f$.

Proof of Lemma (1).

Let A, B, C, D be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$, $h : C \longrightarrow D$ be functions.

Note that $(h \circ g) \circ f$, $h \circ (g \circ f)$ have the same domain, namely, A.

Also note that $(h \circ g) \circ f$, $h \circ (g \circ f)$ have the same range, namely, D.

[We want to verify: For any $x \in A$, $((h \circ g) \circ f)(x) = (h \circ (g \circ f))(x)$.]

Pick any $x \in A$.

$((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$.

$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$.

Then $((h \circ g) \circ f)(x) = (h \circ (g \circ f))(x)$.

It follows that $(h \circ g) \circ f = h \circ (g \circ f)$ as functions. $\square$

6. **Identity function, inclusion function, restrictions and extensions.**

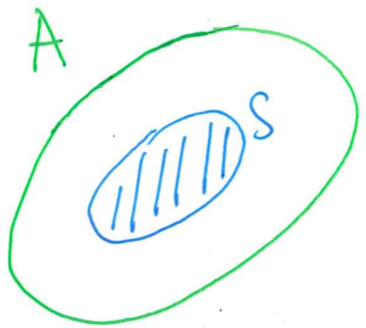
Here are the formal definitions (in terms of set language) of several miscellaneous notions used in various ocassions.

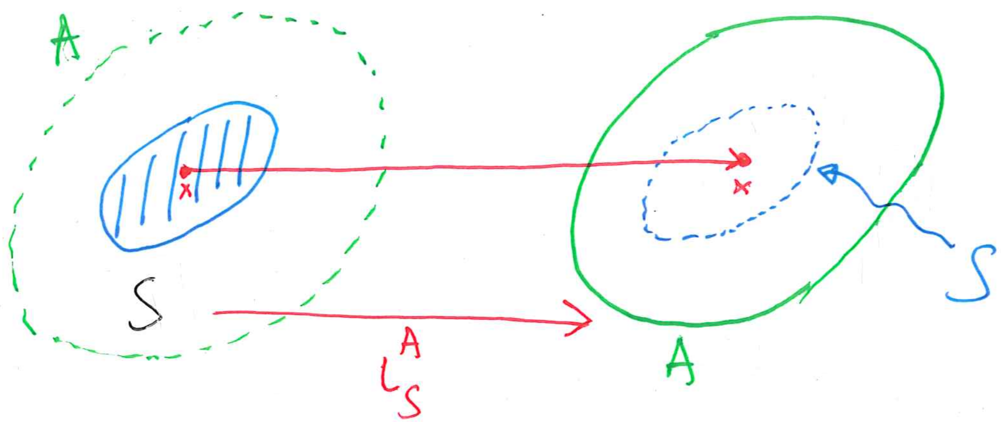
Definition.

Let A be a set.

(a) *Define the function* $\mathsf{id}_A : A \longrightarrow A$ *by* $\mathsf{id}_A(x) = x$ *for any* $x \in A$. id_A *is called* **the identity function on** A.

(b) *Let S be a subset of A. Define the function* $\iota_S^A : S \longrightarrow A$ *by* $\iota_S^A(x) = x$ *for any* $x \in S$. ι_S^A *is called the* **inclusion function of** S **into** A.





Reminder:

In general, ι_S^A is not the same as the identity function on A.

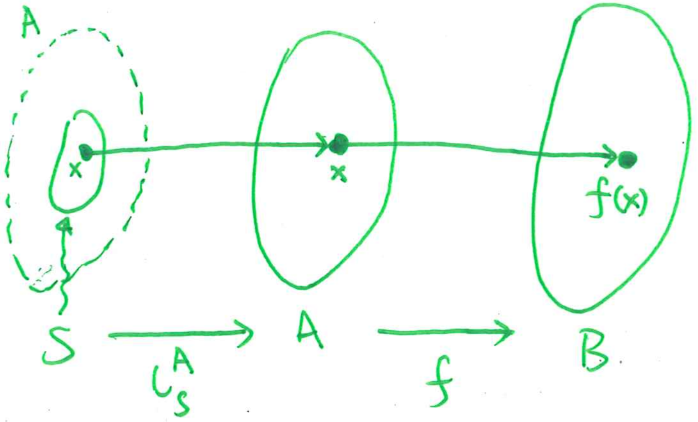
In fact, $\iota_S^A = \mathrm{id}_A$ as functions iff $S = A$.

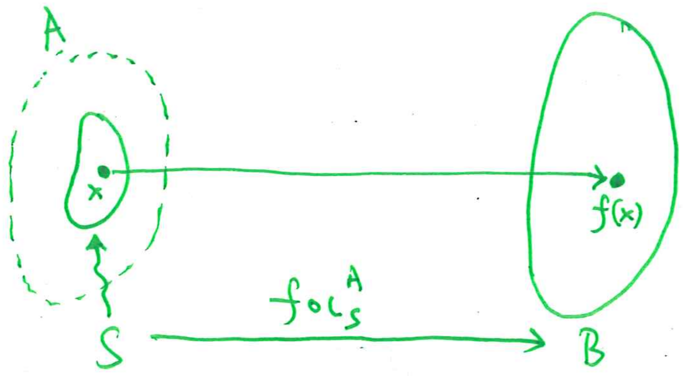
Definition.

Let A, B be sets, and $f : A \longrightarrow B$ be a function.

(a) *Let S be a subset of A. The function $f \circ \iota_S^A : S \longrightarrow B$ is called the* **restriction of f to S**. *It is denoted by $f|_S$.*

(b) *Let H be a set which contains A as a subset, K be a set which contains B as a subset. Suppose $g : H \longrightarrow K$ be a function which satisfies $g \circ \iota_A^H = \iota_B^K \circ f$. Then g is called an* **extension of f**.





Examples of restrictions from one-variable calculus.

① $\tan|_{(-\frac{\pi}{2}, \frac{\pi}{2})} : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$

is the function obtained by 'restricting' the tangent function to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

② $\sin|_{[-\frac{\pi}{2}, \frac{\pi}{2}]} : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$

is the function obtained by 'restricting' the sine function to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Why relevant? Such 'restrictions' are useful when we want to introduce 'arctangent', 'arcsine' respectively.

7. **Graphs of functions.**

To generalize the notion of graphs of functions, we need bring in the notion of cartesian products for two arbitrary sets.

Definition.

Let A, B *be sets, and* $f : A \longrightarrow B$ *be a function. Define* $G = \{(x, f(x)) \mid x \in A\}$*.*
G *is called the* **graph of the function** f*. Note that* $G \subset A \times B$*.*

Lemma (2). (Equality of functions and equality of graphs.)

Let A, B *be sets, and* $f_1, f_2 : A \longrightarrow B$ *be functions. Suppose* G_1, G_2 *are the respective graphs of* f_1, f_2*. Then* f_1 *is equal to* f_2 *as functions iff* $G_1 = G_2$*.*

Proof of Lemma (2). Exercise in set language.

8.. **‘Coordinate plane diagrams’.**

We may visualize a function, displaying its graph, by its ‘**coordinate plane diagram**’.

We illustrate the idea with the example below:

Let $A = \{m, n, p, q, r, s, t, ...\}$, $B = \{c, d, e, g, h, ...\}$, *and* $f : A \longrightarrow B$ *be defined by*

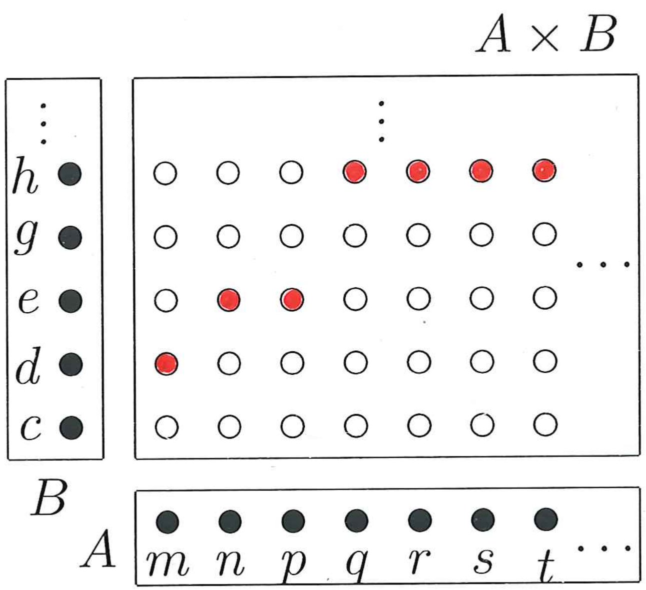
$$f(m) = d,\ f(n) = e,\ f(p) = e,\ f(q) = h,\ f(r) = h,\ f(s) = h,\ f(t) = h, \cdots .$$

By definition, the graph of f *is the set*

$$G = \{(m, d), (n, e), (p, e), (q, h), (r, h), (s, h), (t, h), \cdots\}$$

We draw the ‘coordinate plane diagram’ of the function f *as:*

$(m, d) \in G$
$(n, e) \in G$
$(p, e) \in G$
$(q, h) \in G$
$(r, h) \in G$
$(s, h) \in G$
$(t, h) \in G$
$\vdots$



9. ‘Blobs-and-arrows diagram’ versus ‘coordinate plane diagram’.

Depending on how we like the ‘information’ concerned with a given function $f : A \longrightarrow B$ is presented, we may draw its ‘**coordinate plane diagram**’ or its ‘**blobs-and-arrows diagram**’.

Each has its own advantage.

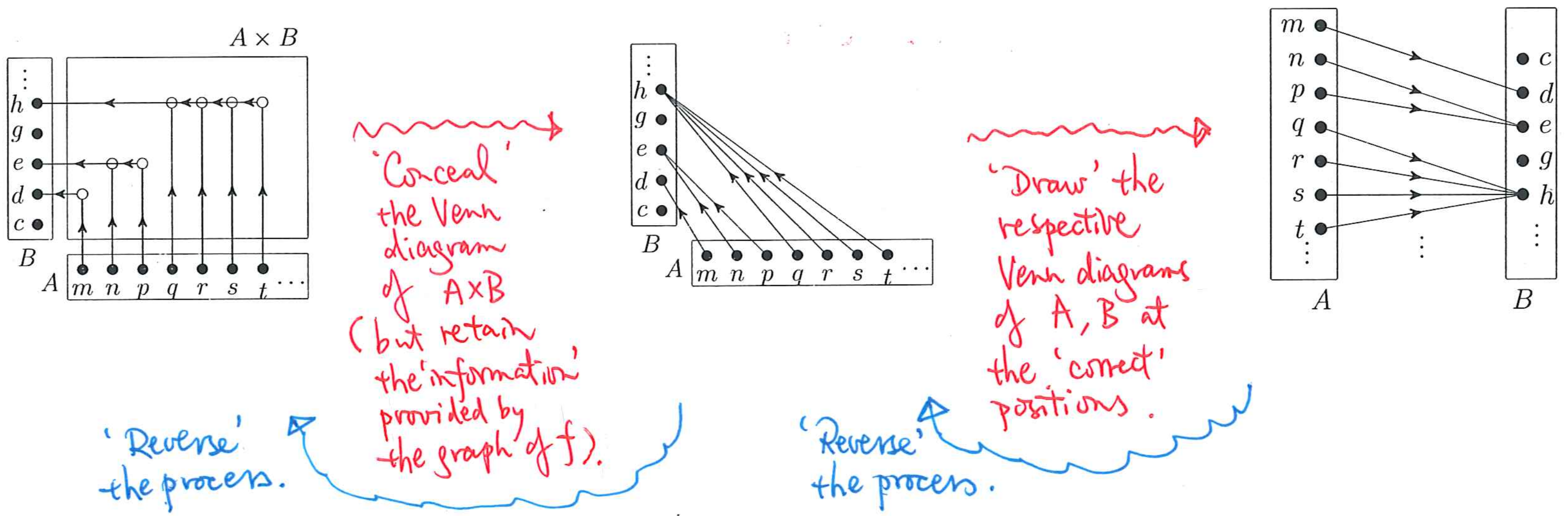
The two diagrams may be converted from one to the other in a systematic way.

Illustration:

Let $A = \{m, n, p, q, r, s, t, ...\}$, $B = \{c, d, e, g, h, ...\}$, *and* $f : A \longrightarrow B$ *be defined by*

$$f(m) = d, f(n) = e, f(p) = e, f(q) = h, f(r) = h, f(s) = h, f(t) = h, \cdots .$$

(a) *‘Coordinate plane diagram’ of f:* (b) *In-between the two kinds of diagrams:* (c) *‘Blobs-and-arrows diagram’ of f:*



10. **Basic examples of functions in school maths and beyond.**

We have encountered various examples of functions in school mathematics and in basic MATH courses.

(a) **Polynomial functions with real coefficients.**

(b) **Rational functions with real coefficients.**

(c) **'Algebraic functions' in school maths.**

(d) **Elementary transcendental functions.**

(e) **'Multivariable functions' in multivariable calculus.**

(f) **Functions of one complex variable.**

(g) **Infinite sequences and families.**

(h) **'Algebraic operations' for algebraic structures.**

(i) **Linear transformations and 'transformation for various algebraic structures'.**

(j) **Various 'operations' in calculus and beyond.**

11. **Families.**

The notion of infinite sequences of real numbers can be generalized to the notion of families for collections of objects.

Definition.

Suppose I, B *are sets, and* $\varphi : I \longrightarrow B$ *is a function.*

Then we say φ *is a* **family in** B**, indexed by** I.

The set I is referred to as the **index set** for this family, and the set
$\{x \in B : x = \varphi(t) \text{ for some } t \in I\}$ is called the **set of all terms** for this family.

Remarks.

(a) This definition generalizes the notion of infinite sequences of real numbers. When $I = \mathbb{N}$, a family in B is just an infinite sequence in B; when furthermore $B = \mathbb{R}$, it is just an infinite sequence of real numbers.

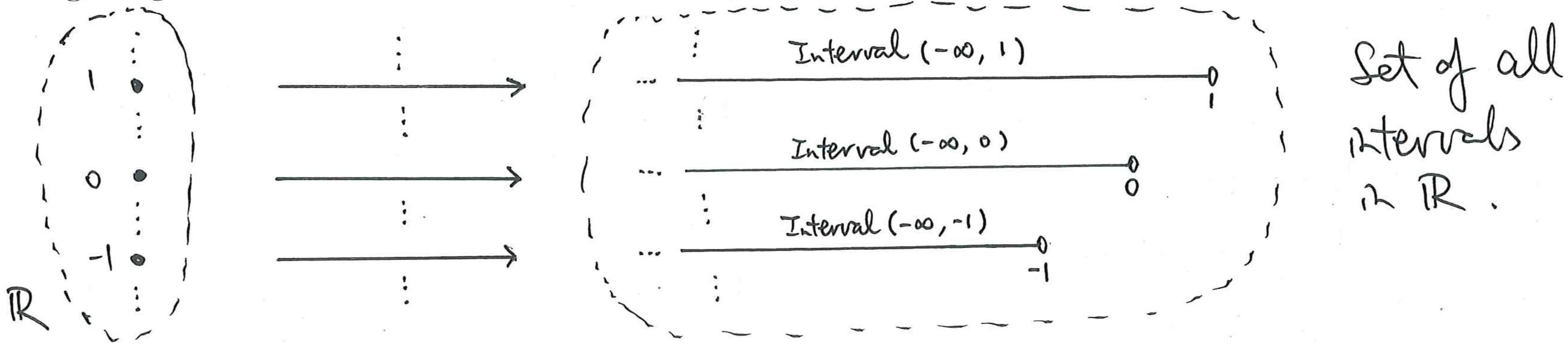
(b) We imitate the notations for infinite sequences when we regard the function φ from I to B as a family in B with index set I. We suppress the symbols φ, B, and present this family as, say, $\{x_\alpha\}_{\alpha \in I}$, in which the symbol x_α stands for $\varphi(\alpha)$ for each $\alpha \in I$.

(c) What is the point of this definition?

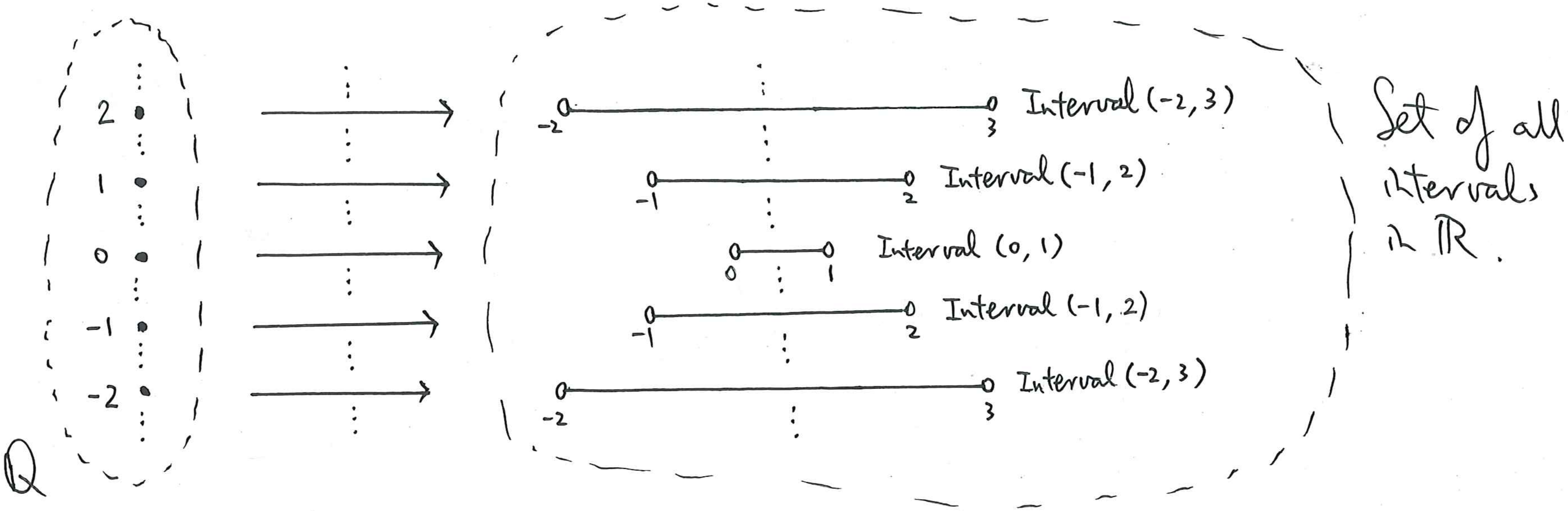
It gives us the flexibility to regard the same family in B, say, $\{x_\alpha\}_{\alpha \in I}$, as a family in C which contains B as a subset, whenever it is convenient for us to do so.

Examples.

(a) $\{(-\infty, u)\}_{u\in\mathbb{R}}$ stands for the function with domain $\mathbb{R}$ and range, say, the set of all intervals, assigning each real number u to the open interval $(-\infty, u)$.



(b) $\{(-|a|, |a|+1)\}_{a\in\mathbb{Q}}$ stands for the function with domain $\mathbb{Q}$ and range, say, the set of all intervals, assigning each rational number a to the open interval $(-|a|, |a|+1)$.



(c) Let p, q be real numbers.

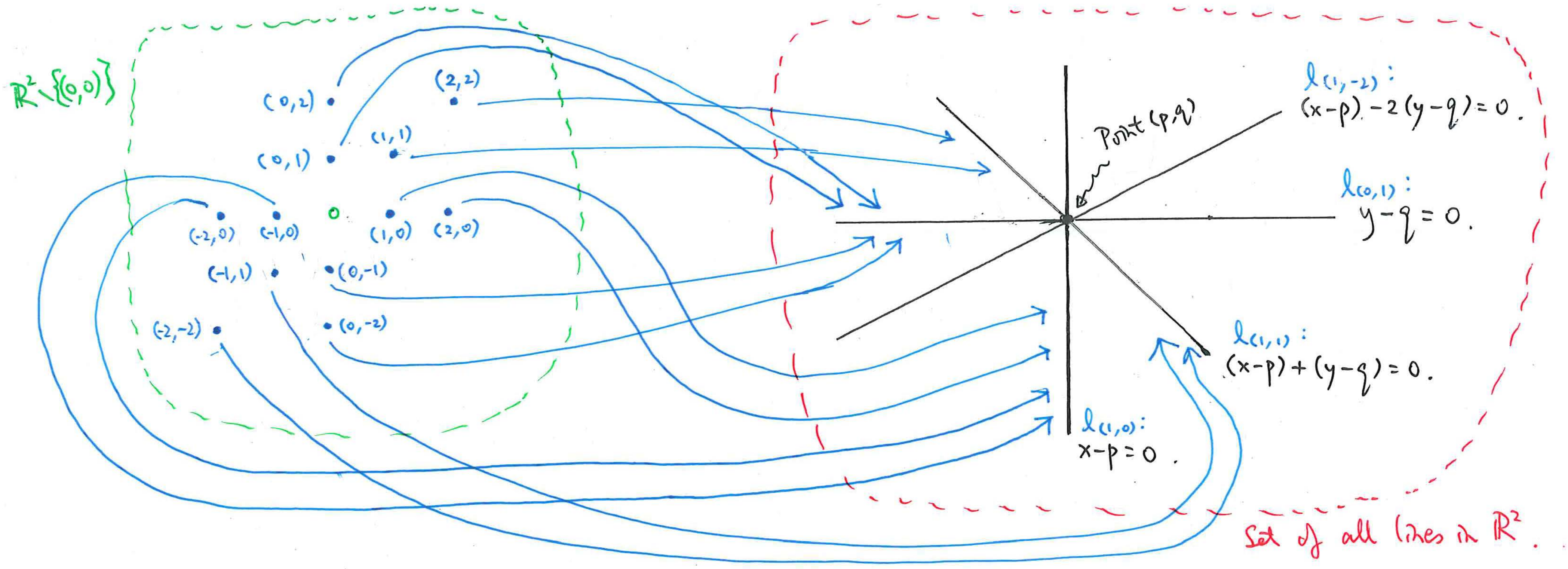
For any real numbers a, b, not both zero, define $\ell_{(a,b)}$ by $\ell_{(a,b)} = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } a(x-p) + b(y-q) = 0\}$.

$\ell_{(a,b)}$ is the line in the infinite plane whose equation is given by $ax + by = ap + bq$.

$\{\ell_{(a,b)}\}_{(a,b)\in\mathbb{R}^2\backslash\{(0,0)\}}$ stands for the function with domain $\mathbb{R}^2\backslash\{(0,0)\}$ and range, say, the set of all lines in the infinite plane, assigning each $(a, b) \in \mathbb{R}^2\backslash\{(0,0)\}$ to the line $\ell_{(a,b)}$.

$\{\ell_{(a,b)}\}_{(a,b)\in\mathbb{R}^2\backslash\{(0,0)\}}$ stands for the function with domain $\mathbb{R}^2\backslash\{(0,0)\}$ and range, say, the set of all lines in the infinite plane, assigning each $(a, b) \in \mathbb{R}^2\backslash\{(0,0)\}$ to the line $\ell_{(a,b)}$.

(The set of all terms of this family is the pencil of all lines passing through (p, q).)



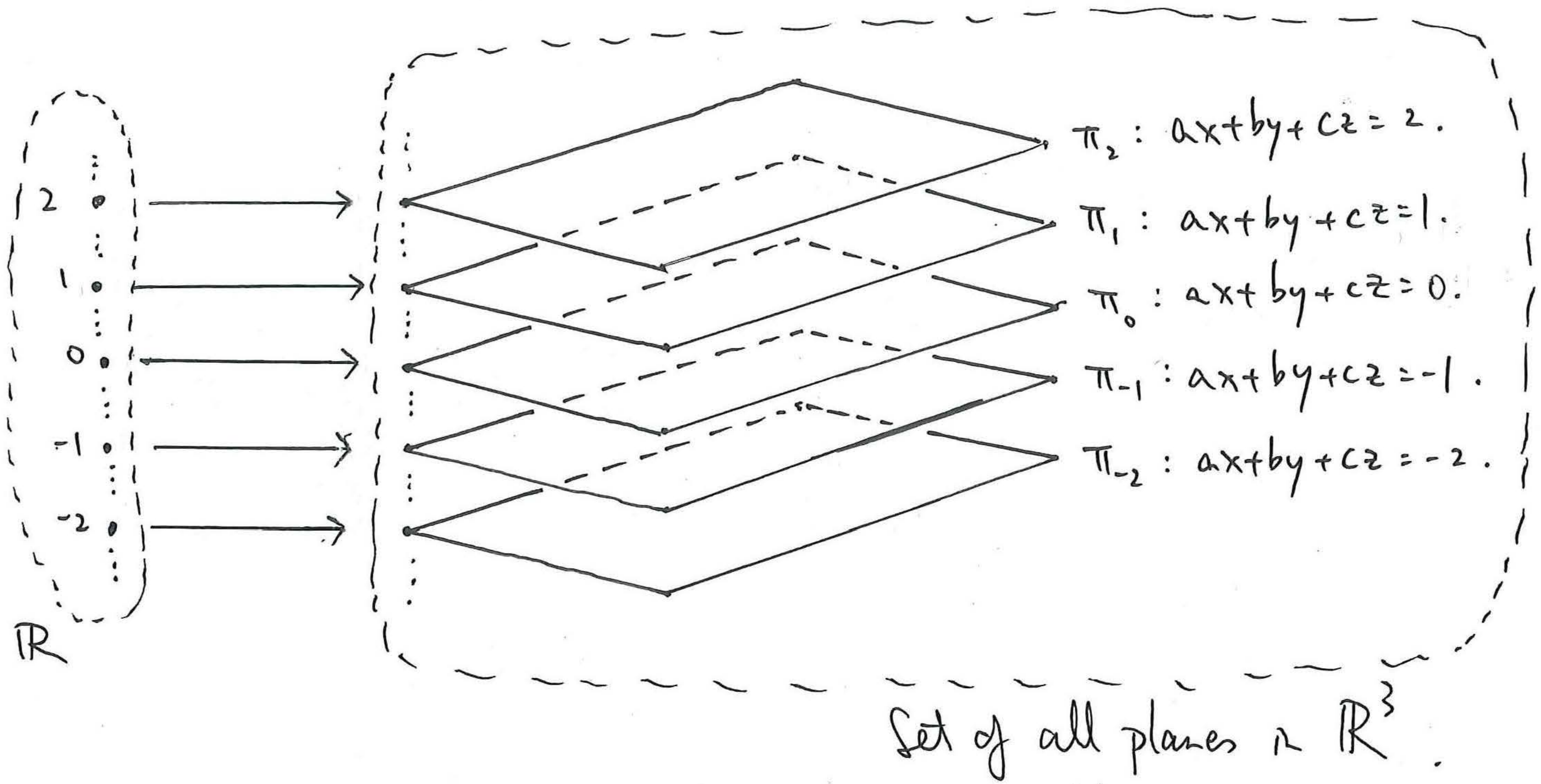
(d) Let a, b, c be real numbers, not all zero.

For any real number d, define π_d by $\pi_d = \{(x, y, z) \mid x, y, z \in \mathbb{R} \text{ and } ax + by + cz = d\}$.

π_d is the plane in the infinite space whose equation is given by $ax + by + cz = d$.

$\{\pi_d\}_{d \in \mathbb{R}}$ stands for the function with domain $\mathbb{R}$ and range, say, the set of all planes in the infinite space, assigning each $d \in \mathbb{R}$ to the plane π_d.

(The set of all terms of this family is the pencil of planes all parallel to the plane $ax + by + cz = 0$.)



(e) For each complex number p, for each positive real number r, define

$$C(\zeta, r) = \{z \in \mathbb{C} : |z - p| = r\}.$$

($C(\zeta, r)$ is the circle in the Argand plane with centre p and radius r.)
Let $p \in \mathbb{C}$.
$\{C(p, r)\}_{r \in (0,+\infty)}$ stands for the function with domain $(0, +\infty)$ and range, say, the set of all circles in the Argand plane, assigning each positive real number to the circle $C(p, r)$.
(The set of all terms of this family is the pencil of circles concentric at p.)

(f) For each complex number p, for each positive real number r, define

$$D(\zeta, r) = \{z \in \mathbb{C} : |z - p| < r\}.$$

($D(\zeta, r)$ is the 'open disc' in the Argand plane with centre p and radius r.)
$\{D(r, r)\}_{r \in (0,+\infty)}$ stands for the function with domain $(0, +\infty)$ and range, say, the set of all open discs in the Argand plane, assigning each positive real number to the open disc $D(r, r)$.

(g) For each real number α, define the polynomial function $f_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ by $f_\alpha(x) = x^2 + \alpha x + 1$ for any $x \in \mathbb{R}$.
$\{f_\alpha\}_{\alpha \in \mathbb{R}}$ stands for the function with domain $\mathbb{R}$ and range, say, the set of all polynomial functions, assigning each real number α to f_α.

12. **Set operations for families of sets.**

The notion of intersection and union for infinite sequences of sets, introduced in the Handout *Universal quantifier and existential quantifier*, can be immediately generalized to families of sets.

Definition.

Let M, I be sets, and $\{S_\alpha\}_{\alpha \in I}$ be a family of subsets of the set M, indexed by I. (For any $\alpha \in I$, S_α is a subset of M.)

(1) *The* **intersection of the family of subsets $\{S_\alpha\}_{\alpha \in I}$ of the set** *M is defined to be the set $\{x \in M : x \in S_\alpha$ for any $\alpha \in I\}$. It is denoted by $\bigcap_{\alpha \in I} S_\alpha$.*

(2) *The* **union of the family of subsets $\{S_\alpha\}_{\alpha \in I}$ of the set** *M is defined to be the set $\{x \in M : x \in S_\alpha$ for some $\alpha \in I\}$. It is denoted by $\bigcup_{\alpha \in I} S_\alpha$.*

Remark. Suppose C is a subset of $\mathfrak{P}(M)$, and we 'index' C by its elements to obtain the family $\{S\}_{S \in C}$. Then the intersection of the set C of subsets of the set M is the intersection of the family $\{S\}_{S \in C}$, and the union of the set C of subsets of the set M is the union of the family $\{S\}_{S \in C}$.

13. Recall this result in the Handout *Universal quantifier and existential quantifier*:

Theorem $(\star)$**.**

Let M be a set and $\{A_n\}_{n=0}^{\infty}$ be an infinite sequence of subsets of M.

(1) *Let S be a subset of M. Suppose $S \subset A_n$ for any $n \in \mathbb{N}$. Then $S \subset \bigcap\limits_{n=0}^{\infty} A_n$.*

(2) *Let S be a subset of M. Suppose $S \subset A_n$ for some $n \in \mathbb{N}$. Then $S \subset \bigcup\limits_{n=0}^{\infty} A_n$.*

(3) *Let T be a subset of M. Suppose $A_n \subset T$ for any $n \in \mathbb{N}$. Then $\bigcup\limits_{n=0}^{\infty} A_n \subset T$.*

(4) *Let T be a subset of M. Suppose $A_n \subset T$ for some $n \in \mathbb{N}$. Then $\bigcap\limits_{n=0}^{\infty} A_n \subset T$.*

(5) *Let C be a subset of M. ($\{A_n \cup C\}_{n=0}^{\infty}, \{A_n \cap C\}_{n=0}^{\infty}, \{A_n \backslash C\}_{n=0}^{\infty}, \{C \backslash A_n\}_{n=0}^{\infty}$ are infinite sequences of subsets of M.) The equalities below hold:*

(5a) $\left(\bigcap\limits_{n=0}^{\infty} A_n\right) \cap C = \bigcap\limits_{n=0}^{\infty} (A_n \cap C)$.

(5b) $\left(\bigcup\limits_{n=0}^{\infty} A_n\right) \cup C = \bigcup\limits_{n=0}^{\infty} (A_n \cup C)$.

(5c) $\left(\bigcap\limits_{n=0}^{\infty} A_n\right) \cup C = \bigcap\limits_{n=0}^{\infty} (A_n \cup C)$.

(5d) $\left(\bigcup\limits_{n=0}^{\infty} A_n\right) \cap C = \bigcup\limits_{n=0}^{\infty} (A_n \cap C)$.

(5e) $\left(\bigcap\limits_{n=0}^{\infty} A_n\right) \backslash C = \bigcap\limits_{n=0}^{\infty} (A_n \backslash C)$.

(5f) $\left(\bigcup\limits_{n=0}^{\infty} A_n\right) \backslash C = \bigcup\limits_{n=0}^{\infty} (A_n \backslash C)$.

(5g) $C \backslash \left(\bigcap\limits_{n=0}^{\infty} A_n\right) = \bigcup\limits_{n=0}^{\infty} (C \backslash A_n)$.

(5h) $C \backslash \left(\bigcup\limits_{n=0}^{\infty} A_n\right) = \bigcap\limits_{n=0}^{\infty} (C \backslash A_n)$.

Theorem $(\star)$ can be generalized immediately to Theorem $(\star')$. The proofs of the respective statements are similar.

Theorem $(\star')$.

Let M, I be sets and $\{A_\alpha\}_{\alpha\in I}$ be a family of subsets of M, indexed by I.

(0) *Suppose $I = \emptyset$. Then $\bigcap\limits_{\alpha\in I} A_\alpha = M$ and $\bigcup\limits_{\alpha\in I} A_\alpha = \emptyset$.*

(1) *Let S be a subset of M. Suppose $S \subset A_\alpha$ for any $\alpha \in I$. Then $S \subset \bigcap\limits_{\alpha\in I} A_\alpha$.*

(2) *Let S be a subset of M. Suppose $S \subset A_\alpha$ for some $\alpha \in I$. Then $S \subset \bigcup\limits_{\alpha\in I} A_\alpha$.*

(3) *Let T be a subset of M. Suppose $A_\alpha \subset T$ for any $\alpha \in I$. Then $\bigcup\limits_{\alpha\in I} A_\alpha \subset T$.*

(4) *Let T be a subset of M. Suppose $A_\alpha \subset T$ for some $\alpha \in I$. Then $\bigcap\limits_{\alpha\in I} A_\alpha \subset T$.*

(5) ...

Theorem ($\star$) can be generalized immediately to Theorem ($\star'$). The proofs of the respective statements are similar.

Theorem ($\star'$).

Let M, I be sets and $\{A_\alpha\}_{\alpha\in I}$ be a family of subsets of M, indexed by I. ...

(5) *Let C be a subset of M. ($\{A_\alpha \cup C\}_{\alpha\in I}, \{A_\alpha \cap C\}_{\alpha\in I}, \{A_\alpha\backslash C\}_{\alpha\in I}, \{C\backslash A_\alpha\}_{\alpha\in I}$ are families of subsets of M.) The equalities below hold:*

(5a) $$\left(\bigcap_{\alpha\in I} A_\alpha\right) \cap C = \bigcap_{\alpha\in I}(A_\alpha \cap C).$$

(5b) $$\left(\bigcup_{\alpha\in I} A_\alpha\right) \cup C = \bigcup_{\alpha\in I}(A_\alpha \cup C).$$

(5c) $$\left(\bigcap_{\alpha\in I} A_\alpha\right) \cup C = \bigcap_{\alpha\in I}(A_\alpha \cup C).$$

(5d) $$\left(\bigcup_{\alpha\in I} A_\alpha\right) \cap C = \bigcup_{\alpha\in I}(A_\alpha \cap C).$$

(5e) $$\left(\bigcap_{\alpha\in I} A_\alpha\right) \backslash C = \bigcap_{\alpha\in I}(A_\alpha \backslash C).$$

(5f) $$\left(\bigcup_{\alpha\in I} A_\alpha\right) \backslash C = \bigcup_{\alpha\in I}(A_\alpha \backslash C).$$

(5g) $$C\backslash\left(\bigcap_{\alpha\in I} A_\alpha\right) = \bigcup_{\alpha\in I}(C\backslash A_\alpha).$$

(5h) $$C\backslash\left(\bigcup_{\alpha\in I} A_\alpha\right) = \bigcap_{\alpha\in I}(C\backslash A_\alpha).$$