1. **Well-ordering Principle for the integers and Least-upper-bound Axiom for the reals.**

   Here we take for granted the validity of two statements, one for the natural number system, the other for the real number system:

   (a) **Well-ordering Principle for the integers (WOPI).**
   Suppose S is a non-empty subset of $\mathbb{N}$. Then S has a least element.

   (b) **Least-upper-bound Axiom for the reals (LUBA).**
   Let A be a non-empty subset of $\mathbb{R}$. Suppose A is bounded above in $\mathbb{R}$. Then A has a supremum in $\mathbb{R}$.

   With the help of the Least-upper-bound Axiom for the reals, we are going to establish the validity of two heuristically obvious statements:

   (a) **Unboundedness of the natural number system in the reals (UNR).**
   $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

   (b) **Archimedean Principle for the reals (AP).**
   For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.

   The Well-ordering Principle for integers will be used later on.

2. **Unboundedness of the natural number system in the reals (UNR).**

   $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

   Proof. [Proof-by-contradiction argument.] Suppose it were true that $\mathbb{N}$ was bounded above in $\mathbb{R}$.

   Note that $0 \in \mathbb{N}$. Then $\mathbb{N} \neq \emptyset$.

   Then, by the Least-upper-bound Axiom, $\mathbb{N}$ would have a supremum in $\mathbb{R}$. We denote this number by σ.

   [*Idea on how to proceed further.* Ask: What is wrong with the existence of such a number σ? Could σ be greater, by a definite amount, say, 0.5, than every natural number? Why (not)? If no, then in light of the presence of some natural number, say, n_0, to be less than σ by at most 0.5, what can we say about the number $n_0 + 1$? Do we have a contradiction now? Why?]

   Write $\varepsilon_0 = \frac{1}{2}$. We have $\sigma - \varepsilon_0 < \sigma$. Then $\sigma - \varepsilon_0$ would not be an upper bound of $\mathbb{N}$ in $\mathbb{R}$. There would exist some $n_0 \in \mathbb{N}$ such that $n_0 > \sigma - \varepsilon_0$.

   Since $n_0 \in \mathbb{N}$, we have $n_0 + 1 \in \mathbb{N}$. Now note that $n_0 + 1 > \sigma - \epsilon_0 + 1 = \sigma + \frac{1}{2} > \sigma$. Then σ would not be an upper bound of $\mathbb{N}$ in $\mathbb{R}$. Contradiction arises.

   Hence $\mathbb{N}$ is not bounded above in $\mathbb{R}$ in the first place.

3. **Archimedean Principle for the reals (AP).**

   For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.

   Proof. Pick any $\varepsilon > 0$.

   [What do we want? Name an appropriate positive integer N which satisfies $N\varepsilon > 1$. So ask: $1 \cdot \varepsilon > 1$? $2\varepsilon > 1$? $3\varepsilon > 1$? ... Or how about $\frac{1}{\varepsilon} < 1$? $\frac{1}{\varepsilon} < 2$? $\frac{1}{\varepsilon} < 3$? ...]

   Note that $\frac{1}{\varepsilon} > 0$. By the unboundedness of $\mathbb{N}$ in $\mathbb{R}$, $\frac{1}{\varepsilon}$ is not an upper bound of $\mathbb{N}$ in $\mathbb{R}$. Then there exists some $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. By definition, $N \in \mathbb{N}\backslash\{0\}$ and $N\varepsilon > 1$.

   Corollary to (AP). (Re-formulations of the Archimedean Principle.)

   Each of the statements below is logically equivalent to each other

   (1) *For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.*

   (2) *For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $\frac{1}{N} < \varepsilon$.*

(3) *For any $K > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N > K$.*

Remark. In fact (UNR) is logically equivalent to (AP). (Proof?)

4. **Dense-ness of the rationals and irrationals in the reals.**

With the help of all the above statements, we establish the validity of some heuristically obvious statements about the rational numbers and irrational numbers.

Theorem (D1). ('Dense-ness' of positive rational numbers amongst positive real numbers.)

Let $\alpha, \beta \in \mathbb{R}$. Suppose $\beta > \alpha > 0$. Then there exists some $r \in \mathbb{Q}$ such that $\alpha < r < \beta$.

Remark. Strictly between any two distinct positive real numbers, there is at least one positive rational number.

Proof. Postponed.

Corollary (D2). ('Dense-ness' of the rationals amongst the reals.)

Let $\alpha, \beta \in \mathbb{R}$. Suppose $\alpha < \beta$. Then there exists some $r \in \mathbb{Q}$ such that $\alpha < r < \beta$.

Remark. Strictly between any two distinct real numbers, there is at least one rational number. Hence there are infinitely many such rational numbers. (Why?)

Proof. Exercise. (Apply Theorem (D1) in various cases.)

Corollary (D3). ('Dense-ness' of the irrationals amongst the reals.)

Let $\alpha, \beta \in \mathbb{R}$. Suppose $\alpha < \beta$. Then there exists some $u \in \mathbb{R}\backslash\mathbb{Q}$ such that $\alpha < u < \beta$.

Remark. Strictly between any two distinct real numbers, there is at least one irrational number. Hence there are infinitely many such irrational numbers. (Why?)

Proof. Let $\alpha, \beta \in \mathbb{R}$. Suppose $\alpha < \beta$. By Corollary (D2), there exists some $s \in \mathbb{Q}$ such that $\alpha < s < \beta$. Again by Corollary (D2), there exists some $t \in \mathbb{Q}$ such that $s < t < \beta$. Now define $u = s + \dfrac{t-s}{\sqrt{2}}$. By definition, u is an irrational number. (Why?) Also, $\alpha < s < u < t < \beta$. (Why?)

The phenomena described in Corollary (2) and Corollary (3) are known as dense-ness in the reals.

Definition. (Dense-ness in the reals.)

Let D be a subset of $\mathbb{R}$. D is said to be **dense in** $\mathbb{R}$ *if every open interval in $\mathbb{R}$ contains some element of D.*

Corollary (D4).

$\mathbb{Q}$ is dense in $\mathbb{R}$, and $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$.

Remark. Not every 'important' subset of $\mathbb{R}$ has such a property: for instance, neither $\mathbb{N}$ nor $\mathbb{Z}$ is dense in $\mathbb{R}$.

5. **Proof of Theorem (1).**

Let $\alpha, \beta \in \mathbb{R}$. Suppose $\beta > \alpha > 0$.

> [Ask. What do we want? Name an appropriate rational number which lies strictly between α and β.]
>
> [*Idea.* Imagine you may choose some positive integer N and then will mark on the postive half-line all the points in $\{k/N \mid k \in \mathbb{N}\}$. Which N will you choose so as to *definitely guarantee* that at least one such point, say, M/N, satisfies $\alpha < M/N < \beta$? This N needs be large, but how large? What if we want $\alpha < (M+1)/N < \beta$ as well?]

Define $\varepsilon = \beta - \alpha$. By definition, $\varepsilon > 0$.

By the Archimedean Principle, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.

Define $S = \left\{ m \in \mathbb{N} : m \cdot \dfrac{1}{N} > \alpha \right\}$.

By the unboundedness of $\mathbb{N}$ in $\mathbb{R}$, there exists some $p \in \mathbb{N}$ such that $p > N\alpha$. Then $p \cdot \dfrac{1}{N} > \alpha$. Therefore $p \in S$. Hence $S \neq \emptyset$.

By the Well-ordering Principle for integers, S has a least element, which we denote by M.

Define $r = \dfrac{M}{N}$. By definition, we have $r \in \mathbb{Q}$, and $r = M \cdot \dfrac{1}{N} > \alpha$.

Also by definition, $M - 1 \notin S$. (Then $M - 1 < 0$ or ($M - 1 \geq 0$ and $(M-1) \cdot \dfrac{1}{N} \leq \alpha$). If $M - 1 < 0$, then we have $(M-1) \cdot \dfrac{1}{N} < 0 \leq \alpha$.) Therefore $(M-1) \cdot \dfrac{1}{N} \leq \alpha$ (in any case).

Now $r = \dfrac{M-1+1}{N} = (M-1)\cdot\dfrac{1}{N} + \dfrac{1}{N} \leq \alpha + \dfrac{1}{N} < \alpha + (\beta - \alpha) = \beta$.

6. **Appendix 1: The motivation in Cantor's construction of the real number system.**

Theorem (IR1).

Let α be an irrational number.

Let $\{r_n\}_{n=0}^{\infty}$ be a strictly decreasing infinite sequence of rational numbers which converges to 0. The statements below hold:

(a) *For any $n \in \mathbb{N}$, there exists some $c_n \in \mathbb{Q}$ such that $\alpha - r_n < c_n < \alpha + r_n$.*

(b) *$\{c_n\}_{n=0}^{\infty}$ is an infinite sequence of rational numbers which converges to α.*

(c) *For any positive rational number q, there exist some $N \in \mathbb{N}$ such that for any $m, n \in \mathbb{N}$, if $m > N$ and $n > N$ then $|c_m - c_n| < q$.*

Statement (a) is an immediate consequence of Corollary (D2).

Statements (b), (c) are consequences of Statement (a) together with the definition for the notion of *limit of sequence*. (For the definition the notion of *limit of sequence*, refer to *Monotonicity and boundedness for infinite sequences of real numbers.*)

Remark. What the above says, in plain words, is that it is possible to approximate the irrational number α as accurately as we like with infinite sequences of rational numbers which converges to α, such as $\{c_n\}_{n=0}^{\infty}$.

The idea that irrational numbers can be approximated as accurately as possible by infinite sequences of rational numbers was exploited by Cantor in his approach in constructing the real number system out of the rational number system.

Definition.

Let $\{c_n\}_{n=0}^{\infty}$ be an infinite sequence of rational numbers. The sequence $\{c_n\}_{n=0}^{\infty}$ is said to be a fundamental sequence if the statement (FS) *holds:*

(FS) *For any positive rational number q, there exist some $N \in \mathbb{N}$ such that for any $m, n \in \mathbb{N}$, if $m > N$ and $n > N$ then $|c_m - c_n| < q$.*

Illustrations.

(a) The infinite sequence of rational numbers $\left\{\displaystyle\sum_{k=0}^{n} \frac{1}{k!}\right\}_{n=0}^{\infty}$ is a fundamental sequence.

It is identified as the irrational number e, according to Cantor.

(b) Let $k \in \mathbb{N}\backslash\{0, 1\}$ and p be a positive prime number.

Define the infinite sequence $\{c_n\}_{n=0}^{\infty}$ recursively by

$$\begin{cases} c_0 &= p \\ c_{n+1} &= \dfrac{1}{k}\left[(k-1)c_n + \dfrac{p}{{c_n}^{k-1}}\right] \quad \text{for any } n \in \mathbb{N} \end{cases}$$

Note that $\{c_n\}_{n=0}^{\infty}$ is an infinite sequence of rational numbers. We can verify that $\{c_n\}_{n=0}^{\infty}$ is a fundamental sequence.

$\{c_n\}_{n=0}^{\infty}$ is identified as the irrational number $\sqrt[n]{p}$, according to Cantor.

Further remark. The result below is along a similar line of thought to Theorem (IR1):

- **Theorem (IR1').**

  Let α be a irrational number.

  Let $\{r_n\}_{n=0}^{\infty}$ be a strictly decreasing infinite sequence of positive rational numbers which converges to 0. The statements below hold:

  (a) *For any $n \in \mathbb{N}$, there exists some $a_n, b_n \in \mathbb{Q}$ such that $\alpha - r_n < a_n < \alpha - r_{n+1} < \alpha < \alpha + r_{n+1} < b_n < \alpha + r_n$.*

  (b) *$\{a_n\}_{n=0}^{\infty}$ is a strictly increasing infinite sequence of rational numbers which converges to α.*

  (c) *$\{b_n\}_{n=0}^{\infty}$ is a strictly decreasing infinite sequence of rational numbers which converges to α.*

The infinite sequence of intervals $\{[a_n, b_n]\}_{n=0}^{\infty}$ is something known as a **nested sequence of interval**, in the sense that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for any $n \in \mathbb{N}$.

It happens that its generalized intersection $\bigcap\limits_{n=0}^{\infty}[a_n, b_n]$ is simply the singleton $\{\alpha\}$.

In this sense, we may say that it is possible to approximate the irrational number α as accurately as we like with an infinite sequence of closed and bounded intervals with rational endpoints which 'will eventually shrink to α', for example $\{[a_n, b_n]\}_{n=0}^{\infty}$.

This point of view is useful in *numerical mathematics*.

7. **Appendix 2: The motivation in Dedekind's construction of the real number system.**

Theorem (IR2).

Let α be an irrational number.

Let $A_\alpha = (-\infty, \alpha) \cap \mathbb{Q}$, $B_\alpha = (\alpha, +\infty) \cap \mathbb{Q}$.

The statements below hold:

(a) ($\star$) *For any $s \in A_\alpha$, for any $t \in B_\alpha$, $s < \alpha < t$.*
  (†) *$A_\alpha \cap B_\alpha = \emptyset$.*
  (‡) *$A_\alpha \cup B_\alpha = \mathbb{Q}$.*

(b) i. *A_α is bounded above in $\mathbb{R}$ by every element of B_α.*
  ii. *A_α has no greatest element.*

(c) i. *B_α is bounded below in $\mathbb{R}$ by every element of A_α.*
  ii. *B_α has no least element.*

(d) i. *The supremum of A_α in $\mathbb{R}$ is α.*
  ii. *The infimum of B_α in $\mathbb{R}$ is α.*

The proofs for Statements ($\star$), (†), (‡) are straightforward exercises in set language and inequalities.

Statements (b.i), (c.i) are immediate consequences of Statement ($\star$).

We prove Statements (b.ii), (d.i) below. (The proofs for Statements (c.ii), (d.ii) are similar.)

Proof of Statement (b.ii).

Suppose it were true that A_α had a greatest element, say, s.

By the definition of A_α, $s < \alpha$. Then, by Corollary (D2), there would exist some $s' \in \mathbb{Q}$ such that $s < s' < \alpha$.

By the definition of A_α, we would have $s' \in A_\alpha$. (Now $s' > s$, and s was by assumption a greatest element of A_α.) Contradiction arises.

Hence A_α has no greatest element in the first place.

Proof of Statement (d.i).

By the definition of A_α, α is an upper bound of A_α in $\mathbb{R}$.

We verify that for any $\beta \in \mathbb{R}$, if β is an upper bound of A_α in $\mathbb{R}$, then $\alpha \leq \beta$:

> Pick any $\beta \in \mathbb{R}$. Suppose β is an upper bound of A_α in $\mathbb{R}$.
> Suppose it were true that $\beta < \alpha$.
> By Corollary (D2), there would exist some $s \in \mathbb{Q}$ such that $\beta < s < \alpha$.
> Then, by the definition of A_α, we would have $s \in A_\alpha$. (Now $s > \beta$, and β was assumed to be an upper bound of A_α.) Contradiction arises.
> It follows that $\alpha \leq \beta$ in the first place.

Hence α is the supremum of A_α in $\mathbb{R}$.

Remark. What the above says, in plain words, is that the irrational number α 'splits' $\mathbb{Q}$ into two 'disjoint' sets of rationals, namely A_α, B_α, every rational in A_α being strictly less than every rational in B_α.

The pair of sets A_α, B_α is called the **Dedekind cut induced by the irrational number** α.

The idea that an arbitrary irrational number corresponds to a Dedekind cut induced by that irrational number was exploited by Dedekind in his approach in constructing the real number system out of the rational number system.

Definition.

Let S, T be non-empty subsets of $\mathbb{Q}$. The pair of sets S, T is called a **Dedekind cut** *if $S \cap T = \emptyset$ and $S \cup T = \mathbb{Q}$ and for any $x \in S$, for any $y \in T$, $x < y$.*

Illustrations.

(a) Define $S = \left\{ x \in \mathbb{Q} : x \leq \sum_{k=0}^{n} \frac{1}{k!} \text{ for some } n \in \mathbb{N} \right\}$, $T = \mathbb{Q} \backslash S$.

The Dedekind cut S, T is identified as the irrational number e, according to Dedekind.

(b) Let p be a positive prime number.

Define $S = \{x \in \mathbb{Q} : x \leq 0 \text{ or } x^n < p\}$, $T = \{x \in \mathbb{Q} : x > 0 \text{ and } x^n > p\}$.

Observe that the sets S, T are defined in terms of rational numbers alone.

The Dedekind cut S, T is identified as the irrational number $\sqrt[n]{p}$, according to Dedekind.

8. **Appendix 3: Decimal representation of real numbers.**

How do we represent real numbers with the help of natural numbers?

One method of representation is **decimal representation**. We have been using it since childhood.

We start with a result which is analogous of Division Algorithm for Natural Numbers (Theorem (DAN) in the Handout *Division Algorithm*):

Theorem (DAR).

Let $x, u \in \mathbb{R}$. Suppose $x \geq 0$ and $u > 0$. Then there exist some unique $q \in \mathbb{N}$, $r \in \mathbb{R}$ such that $x = q \times u + r$ and $0 \leq r < u$.

The existence part of Theorem (DAR) relies on the Archimedean Principle and the Well-ordering Principle for Integers. For its proof, imitate how we start the argument for Theorem (1).

The argument for the uniqueness part of Theorem (DAR) is almost the same as that for Theorem (DAN).

Corollary (DAR1).

Let $x \in \mathbb{R}$. Suppose $x \geq 0$. Then there exist some unique $q \in \mathbb{N}$, $r \in \mathbb{R}$ such that $x = q + r$ and $0 \leq r < 1$.

Remark on terminology. In the context of Corollary (DAR1), We denote the natural number q by $\lfloor x \rfloor$, and call it is called the **integral part of the non-negative real number** x. The number r is referred to as the **non-integral part of the non-negative real number** x.

Definition. (Decimal representation of real numbers between 0 and 1.)

Let $d \in \mathbb{R}$. Suppose $0 \leq d < 1$.

Let $\{d_n\}_{n=0}^{\infty}$ be an infinite sequence in $[\![0, 9]\!]$.

Suppose the infinite sequence $\left\{ \sum_{k=0}^{p} \frac{d_k}{10^{k+1}} \right\}_{p=0}^{\infty}$ converges to d.

Then we say $\left\{ \sum_{k=0}^{p} \frac{d_k}{10^{k+1}} \right\}_{p=0}^{\infty}$ is a **decimal representation of** *d. As a convention, we write $d = 0.d_0 d_1 d_2 d_3 d_4 \cdots$.*

Remarks.

(I) What we actually mean by '$d = 0.d_0 d_1 d_2 d_3 d_4 \cdots$' is '$d = \lim_{p \to \infty} \sum_{k=0}^{p} \frac{d_k}{10^{k+1}}$'. So, for instance, when we write '$\frac{1}{3} = 0.\underbrace{333333\cdots}_{\text{all 3's}}$', what we are actually saying is that $\frac{1}{3}$ is the limit of the infinite sequence $\left\{ \sum_{k=0}^{p} \frac{3}{10^{k+1}} \right\}_{p=0}^{\infty}$.

(II) Some real numbers may admit distinct decimal representations.

For example, $\frac{1}{2} = 0.5\underbrace{000000\cdots}_{\text{all 0's}}$ and $\frac{1}{2} = 0.4\underbrace{999999\cdots}_{\text{all 9's}}$. But this is natural in light of the definition of decimal representation in terms of convergence of infinite sequences.

That every real number between 0 and 1 admits a decimal representation is guaranteed by Theorem (DR).

Theorem (DR).

Let $a \in \mathbb{R}$. Suppose $0 \leq a < 1$.

(a) *For any* $n \in \mathbb{N}$, *define* $\widetilde{a_n} = \lfloor 10^{n+1} a \rfloor$.

$\{\widetilde{a_n}\}_{n=0}^{\infty}$ *is an infinite sequence in* $\mathbb{N}$.

Moreover, $\left\{ \dfrac{\widetilde{a_n}}{10^{n+1}} \right\}_{n=0}^{\infty}$ *is an increasing infinite sequence of real numbers and converges to* a.

(b) *Further define* $a_0 = \widetilde{a_0}$. *For any* $m \in \mathbb{N}\backslash\{0\}$, *further recursively define* $a_m = 10\widetilde{a_{m-1}} - \widetilde{a_m}$.

$\{a_m\}_{m=0}^{\infty}$ *is an infinite sequence in* $[\![0, 9]\!]$.

The infinite sequence $\left\{ \displaystyle\sum_{k=0}^{p} \frac{a_k}{10^{k+1}} \right\}_{p=0}^{\infty}$ *is the same as* $\left\{ \dfrac{\widetilde{a_n}}{10^{n+1}} \right\}_{n=0}^{\infty}$. *It is a decimal representation of* a.

The justification for the convergence of $\left\{ \dfrac{\widetilde{a_n}}{10^{n+1}} \right\}_{n=0}^{\infty}$ to a relies on the formal definition for the notion of *limit of sequence*. The rest of the argument for Theorem (DR) is straightforward.

Illustrations of the ideas in Theorem (DR).

(a) Let $a = \dfrac{1}{3}$.

$\widetilde{a_0} = \left\lfloor \dfrac{10}{3} \right\rfloor = 3$, $\widetilde{a_1} = \left\lfloor \dfrac{100}{3} \right\rfloor = 33$, $\widetilde{a_2} = \left\lfloor \dfrac{1000}{3} \right\rfloor = 333$, et cetera. For each $n \in \mathbb{N}$, $a_n = 3$.

A decimal representation for a is $\left\{ \displaystyle\sum_{j=0}^{n} \frac{3}{10^{j+1}} \right\}_{n=0}^{\infty}$, as expected.

(b) Let $a = \dfrac{1}{5}$.

$\widetilde{a_0} = \left\lfloor \dfrac{10}{5} \right\rfloor = 2$, $\widetilde{a_1} = \left\lfloor \dfrac{100}{5} \right\rfloor = 20$, $\widetilde{a_2} = \left\lfloor \dfrac{1000}{2} \right\rfloor = 200$, et cetera. We have $a_0 = 2$. For each $n \in \mathbb{N}\backslash\{0\}$, $a_n = 0$.

A decimal representation for a is $\left\{ \dfrac{2}{10} + \displaystyle\sum_{j=1}^{n} \frac{0}{10^{j+1}} \right\}_{n=0}^{\infty}$, as expected.

In the light of Theorem (DAR) and Theorem (DR), we may express each non-negative real number x as $x = N.a_0a_1a_2a_3a_4\cdots$, in which N is the integral part of x, and $0.a_0a_1a_2a_3a_4\cdots$ is a decimal representation of the non-integral part of x. We refer to $N.a_0a_1a_2a_3a_4\cdots$ as a **decimal representation of the non-negative real number** x.

When y is a negative real number, $-y$ is a positive real number, and admits a decimal representation $-y = M.b_0b_1b_2b_3b_4\cdots$. We may express y as $y = -M.b_0b_1b_2b_3b_4\cdots$. We refer to $-M.b_0b_1b_2b_3b_4\cdots$ as a **decimal representation of the negative real number** y.