1. **Well-ordering Principle for the integers and Least-upper-bound Axiom for the reals.**

Here we take for granted the validity of these two statements:

(a) **Well-ordering Principle for the integers (WOPI).**

Let S be a non-empty subset of $\mathbb{N}$. S has a least element.

(b) **Least-upper-bound Axiom for the reals (LUBA).**

Let A be a non-empty subset of $\mathbb{R}$. Suppose A is bounded above in $\mathbb{R}$. Then A has a supremum in $\mathbb{R}$.

There is a least element of the set of all upper bounds of A in $\mathbb{R}$. Such a number is called a supremum (least upper bound) of A in $\mathbb{R}$.

With the help of the Least-upper-bound Axiom for the reals, we are going to establish the validity of two heuristically obvious statements:

(a) **Unboundedness of the natural number system in the reals (UNR).**

$\mathbb{N}$ is not bounded above in $\mathbb{R}$.

(b) **Archimedean Principle for the reals (AP).**

For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.

The Well-ordering Principle for integers will be used later on.

2. **Unboundedness of the natural number system in the reals (UNR).**

$\mathbb{N}$ is not bounded above in $\mathbb{R}$.

Proof. [Proof-by-contradiction argument.]

Suppose it were true that $\mathbb{N}$ was bounded above in $\mathbb{R}$.

Note that $0 \in \mathbb{N}$. Then $\mathbb{N} \neq \emptyset$.

Then, by the Least-upper-bound Axiom, $\mathbb{N}$ would have a supremum in $\mathbb{R}$. We denote this number by σ.

[How to proceed further?

- Ask: Is what appears in this picture allowed?

  There is some $\eta > 0$ so that:

$\mathbb{N}$ lies entirely inside $(-\infty, \sigma-\eta]$.

There is absolutely no element of $\mathbb{N}$ inside $(\sigma-\eta, \sigma]$.

- Answer. No; otherwise, we would expect $\sigma-\eta$ to be an upper bound of $\mathbb{N}$ in $\mathbb{R}$, but $\sigma-\eta$ is less than σ.
- Instead, we expect such a picture below:

Now ask: where is n_0+1? Pinpoint n_0+1.]

Take $\varepsilon_0 = \frac{1}{2}$.

$\sigma - \varepsilon_0$ is not an upper bound of $\mathbb{N}$ in $\mathbb{R}$.

[Recall what 'β is an upper bound of S' is.
Recall how to negate this statement.]

Then there would exist some $n_0 \in \mathbb{N}$ such that $n_0 > \sigma - \varepsilon_0$.

Since $n_0 \in \mathbb{N}$, we have $n_0 + 1 \in \mathbb{N}$ also.

Note that $n_0 + 1 > \sigma - \varepsilon_0 + 1 = \sigma + \frac{1}{2} > \sigma$.

But σ was an upper bound of $\mathbb{N}$ in $\mathbb{R}$.

Contradiction arises. $\square$

3. **Archimedean Principle for the reals (AP).**

For any $\varepsilon > 0$, there exists some $N \in \mathbb{N} \backslash \{0\}$ such that $N\varepsilon > 1$.

Proof. Pick any $\varepsilon > 0$.

[What do we want? Name an appropriate positive integer N which satisfies $N\varepsilon > 1$.

So ask: $1 \cdot \varepsilon > 1$? $2\varepsilon > 1$? $3\varepsilon > 1$? ... Or how about $\frac{1}{\varepsilon} < 1$? $\frac{1}{\varepsilon} < 2$? $\frac{1}{\varepsilon} < 3$? ...]

Note that $\frac{1}{\varepsilon} > 0$.

By (UNR), $\frac{1}{\varepsilon}$ is not an upper bound of $\mathbb{N}$ in $\mathbb{R}$. → Recall what 'β is an upper bound of S' is. Then recall how to negate this statement.

Then there exists some $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$.

We have $N > \frac{1}{\varepsilon} > 0$. Then $N \in \mathbb{N} \backslash \{0\}$. Moreover $N\varepsilon > \frac{1}{\varepsilon} \cdot \varepsilon = 1$. $\square$

Corollary to (AP). (Re-formulations of the Archimedean Principle.)

Each of the statements below is logically equivalent to each other

(1) *For any $\varepsilon > 0$, there exists some $N \in \mathbb{N} \backslash \{0\}$ such that $N\varepsilon > 1$.*

(2) *For any $\varepsilon > 0$, there exists some $N \in \mathbb{N} \backslash \{0\}$ such that $\frac{1}{N} < \varepsilon$.*

(3) *For any $K > 0$, there exists some $N \in \mathbb{N} \backslash \{0\}$ such that $N > K$.*

Remark. In fact (UNR) is logically equivalent to (AP). (Proof?)

4. Dense-ness of the rationals and irrationals in the reals.

With the help of all the above statements, we establish the validity of some heuristically obvious statements about the rational numbers and irrational numbers.

Theorem (D1). ('Dense-ness' of positive rational numbers amongst positive real numbers.)

Let $\alpha, \beta \in \mathbb{R}$. Suppose $\beta > \alpha > 0$. Then there exists some $r \in \mathbb{Q}$ such that $\alpha < r < \beta$.

Remark. Strictly between any two distinct positive real numbers, there is at least one positive rational number.

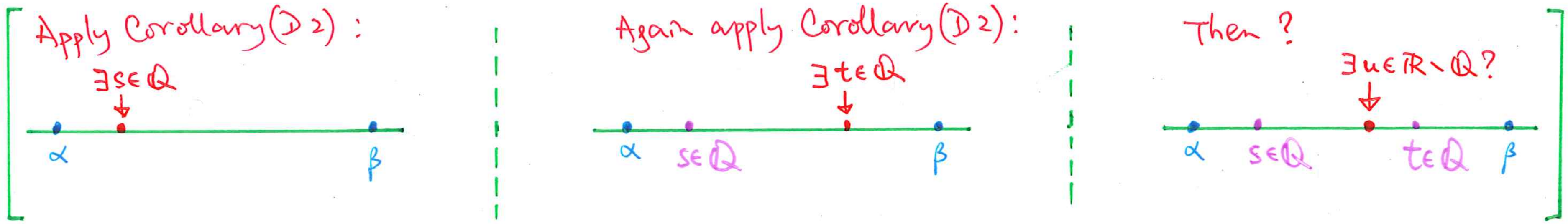
Corollary (D2). ('Dense-ness' of the rationals amongst the reals.)

Let $\alpha, \beta \in \mathbb{R}$. Suppose $\alpha < \beta$. Then there exists some $r \in \mathbb{Q}$ such that $\alpha < r < \beta$.

Corollary (D3). ('Dense-ness' of the irrationals amongst the reals.)

Let $\alpha, \beta \in \mathbb{R}$. Suppose $\alpha < \beta$. Then there exists some $u \in \mathbb{R}\backslash\mathbb{Q}$ such that $\alpha < u < \beta$.

Idea in the argument?



The phenomena described in Corollary (D2) and Corollary (D3) are known as dense-ness in the reals.

Definition. (Dense-ness in the reals.)
Let D be a subset of $\mathbb{R}$.
D is said to be **dense in** $\mathbb{R}$ *if every open interval in $\mathbb{R}$ contains some element of D.*

Corollary (D4).
$\mathbb{Q}$ is dense in $\mathbb{R}$.
$\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$.

Remark. Not every 'important' subset of $\mathbb{R}$ has such a property: for instance, neither $\mathbb{N}$ nor $\mathbb{Z}$ is dense in $\mathbb{R}$.

5. **Proof of Theorem (D1).**

Let $\alpha, \beta \in \mathbb{R}$. Suppose $\beta > \alpha > 0$.

[What do we want? Name an appropriate rational number which lies strictly between α and β.]

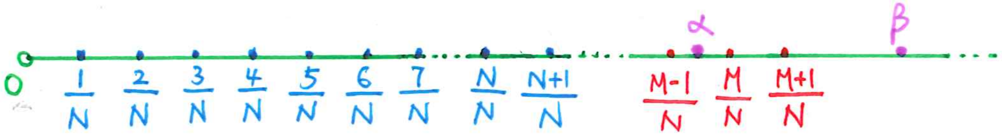
[*Idea.* Imagine you may choose some positive integer N and then will mark on the positive half-line all the points in $\{k/N \mid k \in \mathbb{N}\}$.

Which N will you choose so as to *definitely guarantee* that at least one such point, say, M/N, satisfies $\alpha < M/N < \beta$?

This N needs be large, but how large?

What if we want $\alpha < (M+1)/N < \beta$ as well?]

[Picture:



Observe: It seems that we need $\frac{1}{N} < \beta - \alpha$.]

Define $\varepsilon = \beta - \alpha$. By definition, $\varepsilon > 0$.

By (AP), there exists some $N \in \mathbb{N} \setminus \{0\}$ such that $N\varepsilon > 1$.

Define $S = \left\{ m \in \mathbb{N} : m \cdot \frac{1}{N} > \alpha \right\}$.

We have $S \neq \emptyset$. [Fill in reason: use (UNR).]

By (WOPI), S has a least element, say, M.

Now define $r = \frac{M}{N}$. By definition, $r \in \mathbb{Q}$.

[Ask: Is it true that $\alpha < r < \beta$?]

Since $r = \frac{M}{N}$ and $M \in S$, we have $r > \alpha$.

Also, by the definition of M, we have $M-1 \notin S$.

Then $(M-1) \cdot \frac{1}{N} \leq \alpha$. [Fill in the detail.]

Therefore

$$r = \frac{M}{N} = \frac{(M-1)+1}{N} = \frac{M-1}{N} + \frac{1}{N} < \alpha + \varepsilon = \alpha + (\beta - \alpha) = \beta.$$

$\square$

6. **Appendix 1: The motivation in Cantor's construction of the real number system.**

Theorem (IR1).

Let α be an irrational number.

Let $\{r_n\}_{n=0}^{\infty}$ be a strictly decreasing infinite sequence of rational numbers which converges to 0*. The statements below hold:*

(a) *For any $n \in \mathbb{N}$, there exists some $c_n \in \mathbb{Q}$ such that $\alpha - r_n < c_n < \alpha + r_n$.*

(b) *$\{c_n\}_{n=0}^{\infty}$ is an infinite sequence of rational numbers which converges to α.*

(c) *For any positive rational number q, there exist some $N \in \mathbb{N}$ such that for any $m, n \in \mathbb{N}$, if $m > N$ and $n > N$ then $|c_m - c_n| < q$.*

Statement (a) is an immediate consequence of Corollary (D2).

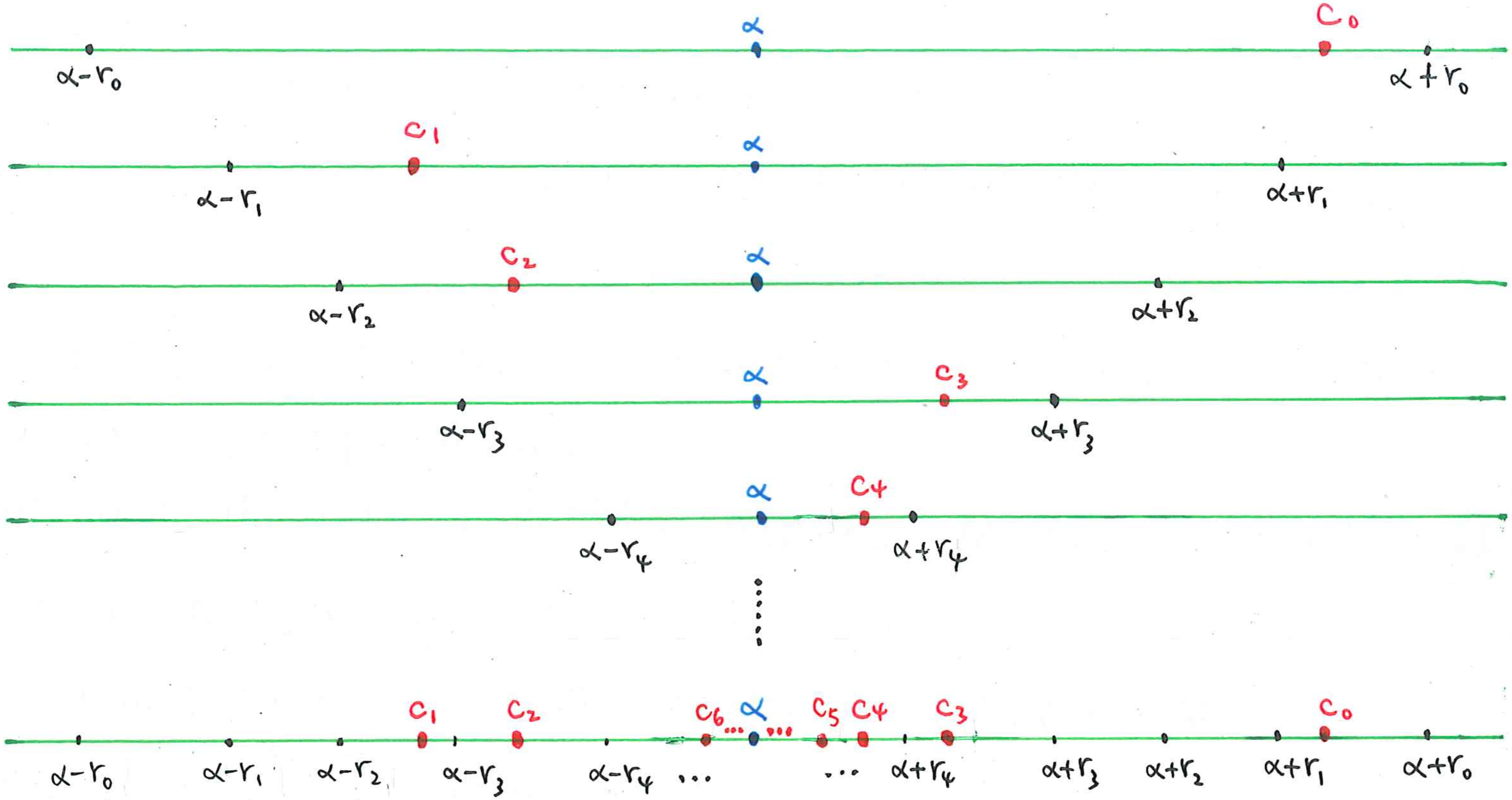
Statements (b), (c) are consequences of Statement (a) together with the definition for the notion of *limit of sequence.*

Remark. What the above says, in plain words, is that it is possible to approximate the irrational number α as accurately as we like with infinite sequences of rational numbers which converges to α, such as $\{c_n\}_{n=0}^{\infty}$.

Visualization of Theorem (IR1).

Assumption.

- α is an irrational number.
- $\{r_n\}_{n=0}^{\infty}$ is a strictly decreasing infinite sequence of rational number.



Conclusion.

(a) For any $n \in \mathbb{N}$, there exists some $c_n \in \mathbb{Q}$ such that $\alpha - r_n < c_n < \alpha + r_n$.

This is a consequence of the dense-ness of $\mathbb{Q}$ in $\mathbb{R}$.

(b) $\{c_n\}_{n=0}^{\infty}$ converges to α. (So α is approximated by $\{c_n\}_{n=0}^{\infty}$ as accurately as we like.)

(c) For any positive rational number q, there exists some $N \in \mathbb{N}$ such that for any $m, n \in \mathbb{N}$, if $m > N$ and $n > N$ then $|c_m - c_n| < q$.

For this reason, $\{c_n\}_{n=0}^{\infty}$ is called a fundamental sequence.

The idea that irrational numbers can be approximated as accurately as possible by infinite sequences of rational numbers was exploited by Cantor in his approach in constructing the real number system out of the rational number system.

Definition.

Let $\{c_n\}_{n=0}^{\infty}$ *be an infinite sequence of rational numbers. The sequence* $\{c_n\}_{n=0}^{\infty}$ *is said to be a* **fundamental sequence** *if the statement* (FS) *holds:*

(FS) *For any positive rational number* q*, there exist some* $N \in \mathbb{N}$ *such that for any* $m, n \in \mathbb{N}$*, if* $m > N$ *and* $n > N$ *then* $|c_m - c_n| < q$.

Illustrations of Cantor's idea in his 'creation' of irrationals out of rationals.

$\sqrt{2}$ is 'created' as the 'fundamental sequence' $\{c_n\}_{n=0}^{\infty}$ recursively defined by

$$\begin{cases} c_0 = 2 \\ c_{k+1} = \frac{1}{2}\left(c_k + \frac{2}{c_k}\right) \text{ for any } k \in \mathbb{N} \end{cases}.$$

e is 'created' as the 'fundamental sequence' $\left\{\sum_{k=0}^{n} \frac{1}{k!}\right\}_{n=0}^{\infty}$.

What is so special about these sequences?

These sequences can be described in terms of rationals alone. From them irrationals are 'created'.

7. Appendix 2: The motivation in Dedekind's construction of the real number system.

Theorem (IR2).

Let α be an irrational number. Let $A_\alpha = (-\infty, \alpha) \cap \mathbb{Q}$, $B_\alpha = (\alpha, +\infty) \cap \mathbb{Q}$.

The statements below hold:

(a) ($\star$) *For any $s \in A_\alpha$, for any $t \in B_\alpha$, $s < \alpha < t$.*

($\dagger$) $A_\alpha \cap B_\alpha = \emptyset$.

($\ddagger$) $A_\alpha \cup B_\alpha = \mathbb{Q}$.

(b) i. *A_α is bounded above in $\mathbb{R}$ by every element of B_α.*

ii. *A_α has no greatest element.*

(c) i. *B_α is bounded below in $\mathbb{R}$ by every element of A_α.*

ii. *B_α has no least element.*

(d) i. *The supremum of A_α in $\mathbb{R}$ is α.*

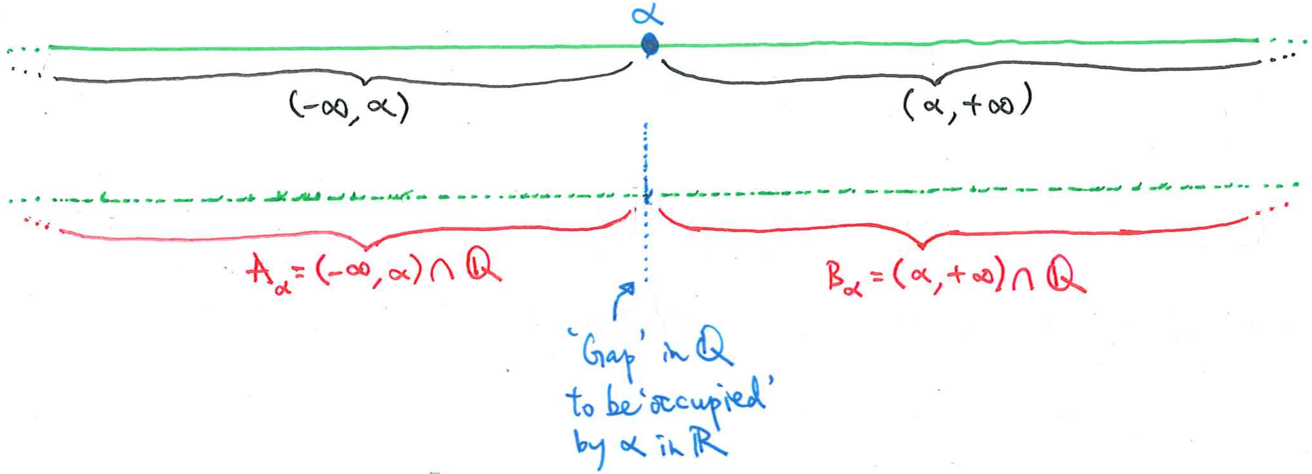
ii. *The infimum of B_α in $\mathbb{R}$ is α.*

The proofs for Statements ($\star$), ($\dagger$), ($\ddagger$) are straightforward exercises in set language and inequalities. Statements (b.i), (c.i) are immediate consequences of Statement ($\star$).

Visualization of Theorem (IR2).

Assumption.

- α is an irrational number.
- $A_\alpha = (-\infty, \alpha) \cap \mathbb{Q}$, $B_\alpha = (\alpha, +\infty) \cap \mathbb{Q}$.



Conclusion.

(a) ($\star$) For any $s \in A_\alpha$, for any $t \in B_\alpha$, $s < \alpha < t$.

($\dagger$) $A_\alpha \cap B_\alpha = \emptyset$.

($\ddagger$) $A_\alpha \cup B_\alpha = \mathbb{Q}$.

[Every point in B_α is strictly to the right of every point in A_α.

α 'splits' the 'rational line' $\mathbb{Q}$ into the 'bi-partition' A_α, B_α.]

(b.i) A_α is bounded above by every element of B_α.

(b.ii) A_α has no greatest element.

(c.i) B_α is bounded below by every element of A_α.

(c.ii) B_α has no least element

(d.i) The supremum of A_α is α.

(d.ii) The infimum of B_α is α.

[α occupies the 'gap' in the 'rational line' $\mathbb{Q}$ which is 'separating' A_α and B_α.]

Remark. What the above says, in plain words, is that the irrational number α 'splits' $\mathbb{Q}$ into two 'disjoint' sets of rationals, namely A_α, B_α, every rational in A_α being strictly less than every rational in B_α.

The pair of sets A_α, B_α is called the **Dedekind cut induced by the irrational number** α.

The idea that an arbitrary irrational number corresponds to a Dedekind cut induced by that irrational number was exploited by Dedekind in his approach in constructing the real number system out of the rational number system.

Definition.

Let S, T be non-empty subsets of $\mathbb{Q}$. The pair of sets S, T is called a **Dedekind cut** *if $S \cap T = \emptyset$ and $S \cup T = \mathbb{Q}$ and for any $x \in S$, for any $y \in T$, $x < y$.*

Illustrations of Dedekind's idea in his 'creation' of irrationals out of rationals.

$\sqrt{2}$ is 'created' as the 'cut' $S = \{x \in \mathbb{Q} : x \leq 0 \text{ or } x^2 < 2\}$, $T = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 2\}$.

e is 'created' as the 'cut' $S = \{x \in \mathbb{Q} : x \leq \sum_{j=0}^{n} \frac{1}{j!} \text{ for some } n \in \mathbb{N}\}$,

$T = \{x \in \mathbb{Q} : x > \sum_{j=0}^{n} \frac{1}{j!} \text{ for any } n \in \mathbb{N}\}$.

What is so special about the sets involved in these 'cuts'?

These sets can be described in terms of rationals alone. From them irrationals are 'created'.

8. Appendix 3: Decimal representation of real numbers.

How do we represent real numbers with the help of natural numbers?

One method of representation is **decimal representation**. We have been using it since childhood.

We start with a result which is analogous of Division Algorithm for Natural Numbers (Theorem (DAN) in the Handout *Division Algorithm*):

Theorem (DAR).

Let $x, u \in \mathbb{R}$. Suppose $x \geq 0$ and $u > 0$.

Then there exist some unique $q \in \mathbb{N}$, $r \in \mathbb{R}$ such that $x = q \times u + r$ and $0 \leq r < u$.

The existence part of Theorem (DAR) relies on the Archimedean Principle and the Well-ordering Principle for Integers. For its proof, imitate how we start the argument for Theorem (1).

The argument for the uniqueness part of Theorem (DAR) is almost the same as that for Theorem (DAN).

Corollary (DAR1).

Let $x \in \mathbb{R}$. Suppose $x \geq 0$.

Then there exist some unique $q \in \mathbb{N}$, $r \in \mathbb{R}$ such that $x = q + r$ and $0 \leq r < 1$.

Remark on terminology. In the context of Corollary (DAR1), We denote the natural number q by $\lfloor x \rfloor$, and call it is called the **integral part of the non-negative real number** x. The number r is referred to as the **non-integral part of the non-negative real number** x.

Definition. (Decimal representation of real numbers between 0 and 1.)

Let $d \in \mathbb{R}$. Suppose $0 \leq d < 1$.

Let $\{d_n\}_{n=0}^{\infty}$ be an infinite sequence in $[\![0, 9]\!]$.

Suppose the infinite sequence $\left\{\sum_{k=0}^{p} \frac{d_k}{10^{k+1}}\right\}_{p=0}^{\infty}$ *converges to d.*

Then we say $\left\{\sum_{k=0}^{p} \frac{d_k}{10^{k+1}}\right\}_{p=0}^{\infty}$ *is a* **decimal representation of** *d.*

As a convention, we write $d = 0.d_0d_1d_2d_3d_4\cdots$.

Remarks.

(I) What we actually mean by '$d = 0.d_0d_1d_2d_3d_4\cdots$' is '$d = \lim_{p\to\infty} \sum_{k=0}^{p} \frac{d_k}{10^{k+1}}$'. So, for instance, when we write '$\frac{1}{3} = 0.\underbrace{333333\cdots}_{\text{all 3's}}$', what we are actually saying is that $\frac{1}{3}$ is the limit of the infinite sequence $\left\{\sum_{k=0}^{p} \frac{3}{10^{k+1}}\right\}_{p=0}^{\infty}$.

(II) Some real numbers may admit distinct decimal representations.

For example, $\frac{1}{2} = 0.5\underbrace{000000\cdots}_{\text{all 0's}}$ and $\frac{1}{2} = 0.4\underbrace{999999\cdots}_{\text{all 9's}}$. But this is natural in light of the definition of decimal representation in terms of convergence of infinite sequences.

That every real number between 0 and 1 admits a decimal representation is guaranteed by Theorem (DR).

Theorem (DR).

Let $a \in \mathbb{R}$. Suppose $0 \leq a < 1$.

(a) *For any $n \in \mathbb{N}$, define $\widetilde{a_n} = \lfloor 10^{n+1}a \rfloor$.*

$\{\widetilde{a_n}\}_{n=0}^{\infty}$ is an infinite sequence in $\mathbb{N}$.

Moreover, $\left\{\dfrac{\widetilde{a_n}}{10^{n+1}}\right\}_{n=0}^{\infty}$ is an increasing infinite sequence of real numbers and converges to a.

(b) *Further define $a_0 = \widetilde{a_0}$. For any $m \in \mathbb{N}\backslash\{0\}$, further recursively define $a_m = 10\widetilde{a_{m-1}} - \widetilde{a_m}$.*

$\{a_m\}_{m=0}^{\infty}$ is an infinite sequence in $[\![0, 9]\!]$.

The infinite sequence $\left\{\displaystyle\sum_{k=0}^{p} \frac{a_k}{10^{k+1}}\right\}_{p=0}^{\infty}$ is the same as $\left\{\dfrac{\widetilde{a_n}}{10^{n+1}}\right\}_{n=0}^{\infty}$. It is a decimal representation of a.

The justification for the convergence of $\left\{\dfrac{\widetilde{a_n}}{10^{n+1}}\right\}_{n=0}^{\infty}$ to a relies on the formal definition for the notion of *limit of sequence.* The rest of the argument for Theorem (DR) is straightforward.

Illustrations of the ideas in Theorem (DR).

(a) Let $a = \dfrac{1}{3}$.

$\widetilde{a_0} = \left\lfloor \dfrac{10}{3} \right\rfloor = 3$, $\widetilde{a_1} = \left\lfloor \dfrac{100}{3} \right\rfloor = 33$, $\widetilde{a_2} = \left\lfloor \dfrac{1000}{3} \right\rfloor = 333$, et cetera.

For each $n \in \mathbb{N}$, $a_n = 3$.

A decimal representation for a is $\left\{ \displaystyle\sum_{j=0}^{n} \frac{3}{10^{j+1}} \right\}_{n=0}^{\infty}$, as expected.

(b) Let $a = \dfrac{1}{5}$.

$\widetilde{a_0} = \left\lfloor \dfrac{10}{5} \right\rfloor = 2$, $\widetilde{a_1} = \left\lfloor \dfrac{100}{5} \right\rfloor = 20$, $\widetilde{a_2} = \left\lfloor \dfrac{1000}{2} \right\rfloor = 200$, et cetera.

We have $a_0 = 2$. For each $n \in \mathbb{N} \backslash \{0\}$, $a_n = 0$.

A decimal representation for a is $\left\{ \dfrac{2}{10} + \displaystyle\sum_{j=1}^{n} \frac{0}{10^{j+1}} \right\}_{n=0}^{\infty}$, as expected.

In the light of Theorem (DAR) and Theorem (DR), we may express each non-negative real number x as

$$x = N.a_0a_1a_2a_3a_4\cdots ,$$

in which N is the integral part of x, and

$$0.a_0a_1a_2a_3a_4\cdots$$

is a decimal representation of the non-integral part of x.

We refer to $N.a_0a_1a_2a_3a_4\cdots$ as a **decimal representation of the non-negative real number** x.

When y is a negative real number, $-y$ is a positive real number, and admits a decimal representation

$$-y = M.b_0b_1b_2b_3b_4\cdots .$$

We may express y as

$$y = -M.b_0b_1b_2b_3b_4\cdots .$$

We refer to $-M.b_0b_1b_2b_3b_4\cdots$ as a **decimal representation of the negative real number** y.