

1. What is the number e ?

You might have been told:

$$e \text{ '=' } \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n, \text{ or } e \text{ '=' } \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!}.$$

But there are two questions:

(A) We are aware that some infinite sequences converge to limits and some do not.

Does $\left\{ \left(1 + \frac{1}{n}\right)^n \right\}_{n=2}^{\infty}$ converge to any limit at all?

Does $\left\{ \sum_{k=0}^n \frac{1}{k!} \right\}_{n=2}^{\infty}$ converge to any limit at all?

(B) Even if both of these infinite sequences converge, do they have the same limit (which, as we have been told, is the number e)?

This is the answer to both Question (A) and Question (B):

Theorem (1).

Let $\{a_n\}_{n=2}^{\infty}$, $\{b_n\}_{n=2}^{\infty}$, $\{c_n\}_{n=2}^{\infty}$ be infinite sequences in \mathbb{R} defined respectively by

$$a_n = \left(1 + \frac{1}{n}\right)^n, \quad b_n = \sum_{k=0}^n \frac{1}{k!}, \quad c_n = \left(1 - \frac{1}{2n}\right) \sum_{k=0}^n \frac{1}{k!} \quad \text{for any } n \in \mathbb{N} \setminus \{0, 1\}.$$

Then $\{a_n\}_{n=2}^{\infty}$, $\{b_n\}_{n=2}^{\infty}$, $\{c_n\}_{n=2}^{\infty}$ converge to the same limit.

Proof of Theorem (1). Postponed; we will use Lemma (2), Lemma (3) and Lemma (4) as ‘stepping stones’.

Remark on notations. From now on, $\{a_n\}_{n=2}^{\infty}$, $\{b_n\}_{n=2}^{\infty}$, $\{c_n\}_{n=2}^{\infty}$ will refer to the same infinite sequences defined in the statement of Theorem (1).

2. Bounded-Monotone Theorem and Sandwich Rule.

The crucial tools used in the proof of Theorem (1) are two results that you have learnt in your *calculus* course.

Bounded-Monotone Theorem (BMT).

Let $\{x_n\}_{n=0}^{\infty}$ be an infinite sequence in \mathbb{R} .

Suppose $\{x_n\}_{n=0}^{\infty}$ is increasing.

Further suppose $\{x_n\}_{n=0}^{\infty}$ is bounded above in \mathbb{R} , (say, by β).

Then $\{x_n\}_{n=0}^{\infty}$ is convergent in \mathbb{R} . (Moreover, $\lim_{n \rightarrow \infty} x_n \leq \beta$.)

Sandwich Rule (SR).

Let $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$ be infinite sequences in \mathbb{R} .

Suppose that for any $n \in \mathbb{N}$, $u_n \leq v_n \leq w_n$.

Further suppose that $\{u_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$ converge to the same limit, say, ℓ in \mathbb{R} .

Then $\{v_n\}_{n=0}^{\infty}$ also converges to ℓ .

Remark. You will learn the proofs of these two results in MATH2050.

3. Lemma (2). (Properties of $\{b_n\}_{n=2}^{\infty}$.)

- (a) $\{b_n\}_{n=2}^{\infty}$ is strictly increasing.
- (b) $\{b_n\}_{n=2}^{\infty}$ is bounded above by 3.
- (c) $\lim_{n \rightarrow \infty} b_n$ exists in \mathbb{R} , and $\lim_{n \rightarrow \infty} b_n \leq 3$.

Remark on notation. For the moment, we write $e_b = \lim_{n \rightarrow \infty} b_n$.

Proof of Lemma (2).

- (a) Let $n \geq 2$. $b_{n+1} - b_n = \sum_{k=0}^{n+1} \frac{1}{k!} - \sum_{k=0}^n \frac{1}{k!} = \frac{1}{(n+1)!} > 0$. Hence $b_{n+1} > b_n$.
 It follows that $\{b_n\}_{n=2}^{\infty}$ is strictly increasing.

- (b) Let $n \geq 2$.

$$b_n = \sum_{k=0}^n \frac{1}{k!} = 2 + \sum_{k=2}^n \frac{1}{k!} = 2 + \sum_{k=2}^n \frac{1}{\underbrace{2 \cdot 3 \cdot 4 \cdots k}_{\text{product of } k-1 \text{ integers each } \geq 2}}$$

$$\leq 2 + \sum_{k=2}^n \frac{1}{2^{k-1}}$$
geometric sum \Rightarrow

$$= 2 + \frac{1}{2} \cdot \frac{1 - 1/2^{n-1}}{1 - 1/2} = 2 + (1 - \frac{1}{2^{n-1}}) < 3$$
 It follows that $\{b_n\}_{n=2}^{\infty}$ is bounded above by 3.

- (c) $\{b_n\}_{n=2}^{\infty}$ is strictly increasing. $\{b_n\}_{n=2}^{\infty}$ is also bounded above by 3.

Then by (BMT), $\{b_n\}_{n=2}^{\infty}$ converges in \mathbb{R} , and $\lim_{n \rightarrow \infty} b_n \leq 3$.

4. Lemma (3). (Properties of $\{a_n\}_{n=2}^\infty$.)

(a) For any $n \in \mathbb{N} \setminus \{0, 1\}$, $a_n = 2 + \sum_{k=2}^n \frac{1}{k!} \cdot 1 \cdot \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) < b_n$.

(b) $\{a_n\}_{n=2}^\infty$ is bounded above by 3.

(c) $\{a_n\}_{n=2}^\infty$ is strictly increasing.

(d) $\lim_{n \rightarrow \infty} a_n$ exists in \mathbb{R} , and $\lim_{n \rightarrow \infty} a_n \leq 3$.

Remark on notation. For the moment, we write $e_a = \lim_{n \rightarrow \infty} a_n$.

Proof of Statement (d) in Lemma (3).

(d) (Suppose we have already proved (a), (b), (c).)

$\{a_n\}_{n=2}^\infty$ is strictly increasing.

$\{a_n\}_{n=2}^\infty$ is also bounded above by 3.

Then by (BMT), $\{a_n\}_{n=2}^\infty$ converges in \mathbb{R} , and $\lim_{n \rightarrow \infty} a_n \leq 3$.

Lemma (3). (Properties of $\{a_n\}_{n=2}^{\infty}$.)

(a) For any $n \in \mathbb{N} \setminus \{0, 1\}$, $a_n = 2 + \sum_{k=2}^n \frac{1}{k!} \cdot 1 \cdot \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) < b_n$.

(b) $\{a_n\}_{n=2}^{\infty}$ is bounded above by 3.

Proof of Statement (a), Statement (b) in Lemma (3).

(a) Let $n \geq 2$.

$$\begin{aligned}
 a_n &= \left(1 + \frac{1}{n}\right)^n \stackrel{\text{Binomial Theorem}}{=} \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{n}\right)^k \\
 &= 1 + n \cdot \frac{1}{n} + \sum_{k=2}^n \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k!} \cdot \frac{1}{n^k} \\
 &= 2 + \sum_{k=2}^n \frac{1}{k!} \cdot \underbrace{1 \cdot \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)}_{\text{product of } k-1 \text{ real numbers each strictly between 0 and 1}} \\
 &< 2 + \sum_{k=2}^n \frac{1}{k!} \cdot 1^k \\
 &= 2 + \sum_{k=2}^n \frac{1}{k!} = \sum_{k=0}^n \frac{1}{k!} = b_n.
 \end{aligned}$$

product of k integers

product of $k-1$ real numbers each strictly between 0 and 1

(b) Let $n \geq 2$. $a_n < b_n < 3$. Therefore $\{a_n\}_{n=2}^{\infty}$ is bounded above by 3.

Lemma (3). (Properties of $\{a_n\}_{n=2}^{\infty}$.)

(c) $\{a_n\}_{n=2}^{\infty}$ is strictly increasing.

Proof of Statement (c) in Lemma (3).

(c) Let $n \geq 2$. Note that $a_n > 0$ and $a_{n+1} > 0$.

$$\frac{a_{n+1}}{a_n} = \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^n} = \left(1 + \frac{1}{n+1}\right) \cdot \frac{\left(1 + \frac{1}{n+1}\right)^n}{\left(1 + \frac{1}{n}\right)^n} = \left(1 + \frac{1}{n+1}\right) \left[\frac{1 + 1/(n+1)}{1 + 1/n}\right]^n$$

$$= \left(1 + \frac{1}{n+1}\right) \left[\frac{(n+2)n}{(n+1)^2}\right]^n$$

$$= \left(1 + \frac{1}{n+1}\right) \left[1 - \frac{1}{(n+1)^2}\right]^n$$

$$> \left(1 + \frac{1}{n+1}\right) \left[1 - \frac{n}{(n+1)^2}\right]$$

$$= \frac{n+2}{n+1} \cdot \frac{(n+1)^2 - n}{(n+1)^2}$$

$$= \frac{n^3 + 3n^2 + 3n + 2}{(n+1)^3} = 1 + \frac{1}{(n+1)^3} > 1.$$

Bernoulli's Inequality:
Whenever $m \in \mathbb{N}$ or $\beta \in [0, 1]$
and $-1 < \beta < 0$,
we have
 $(1 + \beta)^m > 1 + m\beta$.

Therefore $a_{n+1} > a_n$.
It follows that $\{a_n\}_{n=2}^{\infty}$ is strictly increasing.

5. Lemma (4). (Properties of $\{c_n\}_{n=2}^\infty$.)

(a) For any $n \in \mathbb{N} \setminus \{0, 1, 2, 3\}$, $c_n < a_n < b_n$.

Proof of Statement (a) in Lemma (4).

(a) Let $n \geq 4$. We have already proved $a_n < b_n$.

Lemma (3) $\rightarrow a_n \stackrel{\text{Lemma (3)}}{=} 2 + \sum_{k=2}^n \frac{1}{k!} \cdot 1 \cdot \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)$

$\geq 2 + \sum_{k=2}^n \frac{1}{k!} \cdot 1 \cdot \left[1 - \left(\frac{1}{n} + \frac{2}{n} + \cdots + \frac{k-1}{n}\right)\right]$

arithmetic sum $\rightarrow \stackrel{\text{arithmetic sum}}{=} 2 + \sum_{k=2}^n \frac{1}{k!} \cdot \left[1 - \frac{k(k-1)}{2n}\right]$

This is b_n . $\rightarrow = 2 + \sum_{k=2}^n \frac{1}{k!} - \frac{1}{2n} \sum_{k=2}^n \frac{k(k-1)}{k!}$

$= b_n - \frac{1}{2n} \sum_{k=2}^n \frac{1}{(k-2)!} = b_n - \frac{1}{2n} \sum_{j=0}^{n-2} \frac{1}{j!} = b_n - \frac{1}{2n} b_{n-2}$

Lemma (2) $\rightarrow \stackrel{\text{Lemma (2)}}{>} b_n - \frac{1}{2n} b_n$

$= \left(1 - \frac{1}{2n}\right) b_n = c_n$

Hence $c_n < a_n < b_n$.

Weierstrass' Product Inequality:
Whenever $\gamma_1, \gamma_2, \dots, \gamma_m \in (0, 1)$,
the inequality
 $(1-\gamma_1)(1-\gamma_2) \cdots (1-\gamma_m)$
 $> 1 - (\gamma_1 + \gamma_2 + \cdots + \gamma_m)$
holds.

Same expression
with different
dummy indices

Lemma (4). (Properties of $\{c_n\}_{n=2}^{\infty}$.)

(a) For any $n \in \mathbb{N} \setminus \{0, 1, 2, 3\}$, $c_n < a_n < b_n$.

(b) $\lim_{n \rightarrow \infty} c_n$ exists, and $\lim_{n \rightarrow \infty} c_n = e_b$.

Proof of Statement (b) in Lemma (4).

(b) $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{2n}\right)$ exists and is 1.

Also, $\lim_{n \rightarrow \infty} b_n$ exists and is e_b .

Then $\lim_{n \rightarrow \infty} c_n$ exists and is $1 \cdot e_b = e_b$.

6. Proof of Theorem (1).

By Lemma (4), for any $n \in \mathbb{N} \setminus \{0, 1, 2, 3\}$, $c_n < a_n < b_n$.

By Lemma (2), Lemma (3) and Lemma (4), the limits $\lim_{n \rightarrow \infty} a_n$, $\lim_{n \rightarrow \infty} b_n$, $\lim_{n \rightarrow \infty} c_n$ exist.

Their respective values are e_a , e_b , e_b .

Then by (SR), we have $e_b \leq e_a \leq e_b$. Hence $e_a = e_b$.

7. Definition. (The number e .)

We define the real number e to be the common value of the limits

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n$$

and

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!}.$$

Remark. The value of e is $2.718281828459 \dots$.

$$e \stackrel{\text{definition}}{=} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \stackrel{\text{proved}}{=} \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!}$$

$$e^2 = \left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \right]^2 = \left(\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!} \right)^2 \quad (\text{Nothing wrong!})$$

How about $e^2 \stackrel{\text{proof?}}{=} \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n \stackrel{\text{proof?}}{=} \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{2^k}{k!} \quad ???$

$$e^3 = \left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \right]^3 = \left(\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!} \right)^3 \quad (\text{Nothing wrong!})$$

How about $e^3 \stackrel{\text{proof?}}{=} \lim_{n \rightarrow \infty} \left(1 + \frac{3}{n}\right)^n \stackrel{\text{proof?}}{=} \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{3^k}{k!} \quad ???$

8. Appendix 1: Beyond the number e and towards the definition of the exponential function.

You might have been told ‘ $\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n = e^2$ ’, ‘ $\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{3^k}{k!} = e^3$ ’, et cetera.

What ‘ $\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n = e^2$ ’ telling you is :

The infinite sequence $\left\{ \left(1 + \frac{2}{n}\right)^n \right\}_{n=2}^{\infty}$ converges in \mathbb{R} , and its limit is equal to e^2 (which is the square of the number e as we have defined).

But something seems to be wrong:

Even though $\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n$ may exist, it is not immediately apparent why the equality

$$\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{2n}$$

should hold.

You might have been told ‘ $\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n = e^2$ ’, ‘ $\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{3^k}{k!} = e^3$ ’, et cetera. ...

As for ‘ $\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{3^k}{k!} = e^3$ ’, what it is tell you is:

The infinite sequence $\left\{ \sum_{k=0}^n \frac{3^k}{k!} \right\}_{n=2}^{\infty}$ converges in \mathbb{R} , and its limit is equal to e^3 (which is the cube of the number e as we have defined).

Again something seems to be wrong:

Even though $\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{3^k}{k!}$ may exist, it is not immediately apparent why the equality

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{3^k}{k!} = \lim_{n \rightarrow \infty} \left(\sum_{k=0}^n \frac{1}{k!} \right)^3$$

should hold.

So why are these statements true? They are the consequences of Theorem (5), Theorem (6) and Corollary (7).

Theorem (5).

Let α be a positive real number. Let $\{a_n\}_{n=2}^{\infty}$, $\{b_n\}_{n=2}^{\infty}$, $\{c_n\}_{n=2}^{\infty}$ be infinite sequences in \mathbb{R} defined respectively by

$$a_n = \left(1 + \frac{\alpha}{n}\right)^n, \quad b_n = \sum_{k=0}^n \frac{\alpha^k}{k!}, \quad c_n = \left(1 - \frac{\alpha^2}{2n}\right) \sum_{k=0}^n \frac{\alpha^k}{k!} \quad \text{for any } n \in \mathbb{N} \setminus \{0, 1\}.$$

Then $\{a_n\}_{n=2}^{\infty}$, $\{b_n\}_{n=2}^{\infty}$, $\{c_n\}_{n=2}^{\infty}$ converge to the same limit.

Theorem (6).

Let σ, τ be positive real numbers. Define $u_n = \sum_{k=0}^n \frac{\sigma^k}{k!}$, $v_n = \sum_{k=0}^n \frac{\tau^k}{k!}$, $w_n = \sum_{k=0}^n \frac{(\sigma + \tau)^k}{k!}$

for each $n \in \mathbb{N}$. The statements below hold:

- (a) For any $n \in \mathbb{N}$, the inequality $w_n \leq u_n v_n \leq w_{2n}$ holds.
- (b) $\lim_{n \rightarrow \infty} w_n = \left(\lim_{n \rightarrow \infty} u_n\right) \left(\lim_{n \rightarrow \infty} v_n\right)$.

Corollary (7).

For any positive integer α , the equality $\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{\alpha^k}{k!} = e^\alpha$ holds.

$$e \stackrel{\text{definition}}{=} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \stackrel{\text{proved}}{=} \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!}$$

$$e^{-1} = \frac{1}{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n} = \frac{1}{\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!}} \quad (\text{Nothing wrong!})$$

How about $e^{-1} \stackrel{\text{proof?}}{=} \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n \stackrel{\text{proof?}}{=} \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(-1)^k}{k!} \quad ???$

$$e^2 = \left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \right]^2 = \left(\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!} \right)^2 \quad (\text{Nothing wrong!})$$

How about $e^2 \stackrel{\text{proof?}}{=} \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n \stackrel{\text{proof?}}{=} \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{2^k}{k!} \quad ???$

$$e \stackrel{\text{definition}}{=} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \stackrel{\text{proved}}{=} \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!}$$

How about $\sqrt{e} \stackrel{\text{proof?}}{=} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2n}\right)^n \stackrel{\text{proof?}}{=} \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{2^k \cdot (k!)} \quad ???$

$$e^{\sqrt{2}} = \lim_{n \rightarrow \infty} \left(1 + \frac{\sqrt{2}}{n}\right)^n = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(\sqrt{2})^k}{k!} \quad ??????$$

$$e^{\pi} = \lim_{n \rightarrow \infty} \left(1 + \frac{\pi}{n}\right)^n = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{\pi^k}{k!} \quad ??????$$

$$e^e = \lim_{n \rightarrow \infty} \left(1 + \frac{e}{n}\right)^n = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{e^k}{k!} \quad ??????$$

Proofs for these equalities?

But what are $e^{\sqrt{2}}$, e^{π} , e^e really? What is 'u'?

9. Appendix 2: From the exponential function to ‘powers’ and ‘index laws’.

Theorem (5) and Theorem (6) are the first steps towards making sense of the exponential function $\exp : \mathbb{R} \longrightarrow \mathbb{R}$.

After more work, we can prove:

Theorem (8).

(a) For any $\alpha \in \mathbb{R}$, the limits $\lim_{n \rightarrow \infty} \left(1 + \frac{\alpha}{n}\right)^n$, $\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{\alpha^k}{k!}$ exist in \mathbb{R} and are equal to each other.

(b) For any $\alpha, \beta \in \mathbb{R}$, the equality $\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(\alpha + \beta)^k}{k!} = \left(\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{\alpha^k}{k!} \right) \left(\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{\beta^k}{k!} \right)$ holds.

Theorem (8) justifies the definition of the exponential function, and yields Theorem (9), which gives the basic (arithmetic) properties of the exponential function.

Definition. (The exponential function.)

Define the function $\exp : \mathbb{R} \longrightarrow \mathbb{R}$ by $\exp(x) = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{x^k}{k!}$ for any $x \in \mathbb{R}$.

\exp is called the **exponential function** (on the reals).

Theorem (9).

The statements below hold:

(a) $\exp(0) = 1$, and $\exp(1) = e$.

(b) For any $s, t \in \mathbb{R}$, $\exp(s + t) = \exp(s) \exp(t)$.

(c) For any $s \in \mathbb{R}$, $\exp(s) > 0$ and $\exp(-s) = \frac{1}{\exp(s)}$.

Arbitrary real powers of e is in fact defined through the use of the exponential function.

Definition.

For any $\sigma \in \mathbb{R}$, we define the number e^σ by $e^\sigma = \exp(\sigma)$.

Remark. Theorem (9) immediately translates as:

- (a) $e^0 = 1$, and $e^1 = e$.
- (b) For any $s, t \in \mathbb{R}$, $e^{s+t} = e^s e^t$.
- (c) For any $s \in \mathbb{R}$, $e^s > 0$ and $e^{-s} = \frac{1}{e^s}$.

You may wonder what the point of this is.

You may want to ask:

‘Didn’t we know that $e^{s+t} = e^s e^t$ for any real numbers s, t from school maths?’

The answer to this question is:

‘In fact, in school maths we were told $e^{s+t} = e^s e^t$ for any real numbers s, t , but it was not explained why it would be so.

Actually it was not explained why, for instance, $2^{\sqrt{2}+\sqrt{3}} = 2^{\sqrt{2}} \cdot 2^{\sqrt{3}}$ holds. We were not given the explanation because, in the first place, we did not know what $2^{\sqrt{2}}$ was.’

With the help of the exponential function $\exp : \mathbb{R} \longrightarrow \mathbb{R}$ and the notion of *inverse function*, we may make sense of the (natural) logarithmic function $\ln : (0, +\infty) \longrightarrow \mathbb{R}$ that we encountered in school maths. Through the exponential function and the logarithmic function we may make sense of the notion of *arbitrary real powers of arbitrary positive real numbers*, by giving an appropriate definition for them, and justify the ‘index laws’ for them with reference to the definition.

Definition.

Let a be a positive real number, and σ be a real number. We define the number a^σ by $a^\sigma = \exp(\sigma \ln(a))$.

Index Laws.

The statements below hold:

- (a) For any $a > 0$, $a^0 = 1$ and $a^1 = a$.
- (b) For any $a > 0$, for any $\sigma, \tau \in \mathbb{R}$, $a^{\sigma+\tau} = a^\sigma a^\tau$.
- (c) For any $a > 0$, for any $\sigma \in \mathbb{R}$, $a^\sigma > 0$ and $a^{-\sigma} = \frac{1}{a^\sigma}$.

A full treatment of the above will be given in your *analysis* course.