MATH1050 Arithmetico-geometric Inequality

1. **Definitions. (Arithmetic mean, geometric mean and harmonic mean.)**
Let $n \in \mathbb{N}\backslash\{0\}$. Let $a_1, a_2, \cdots, a_n$ be n positive real numbers.
   (a) *The number* $\dfrac{a_1 + a_2 + \cdots + a_n}{n}$ *is called the* **arithmetic mean** *of $a_1, a_2, \cdots, a_n$.*
   (b) *The number* $\sqrt[n]{a_1 a_2 \cdot \ldots \cdot a_n}$ *is called the* **geometric mean** *of $a_1, a_2, \cdots, a_n$.*
   (c) *The number* $\left[\dfrac{1}{n}\left(\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_n}\right)\right]^{-1}$ *is called the* **harmonic mean** *of $a_1, a_2, \cdots, a_n$.*

   Remark. By definition, the harmonic mean of $a_1, a_2, \cdots, a_n$ is the reciprocal of the arithmetic mean of the reciprocals of $a_1, a_2, \cdots, a_n$.

2. **Theorem (1). (Arithmetico-geometrical Inequality.)**
Let $m \in \mathbb{N}\backslash\{0\}$. Let $a_1, a_2, \cdots, a_m$ be m posiitve real numbers.
The inequality $\dfrac{a_1 + a_2 + \cdots + a_m}{m} \geq \sqrt[m]{a_1 a_2 \cdot \ldots \cdot a_m}$ *holds. Equality holds iff $a_1 = a_2 = \cdots = a_m$.*
Remark. So the result says: the arithmetic mean of an arbitrary collection of 'finitely many' positive real numbers is no less than the geometric mean of the same collection. There are many proofs for this result: one of them is given below, as a consequence of Lemma(3), Lemma (4), Lemma (5) and Lemma (6) together.

3. **Corollary (2).**
Let $m \in \mathbb{N}\backslash\{0\}$. Let $a_1, a_2, \cdots, a_m$ be m positive real numbers.
The inequality
$$\left[\frac{1}{m}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_m}\right)\right]^{-1} \leq \sqrt[m]{a_1 a_2 \cdot \ldots \cdot a_m} \leq \frac{a_1 + a_2 + \cdots + a_m}{m}$$
holds. Each equality holds iff $a_1 = a_2 = \cdots = a_m$.
Remark. The proof is left as an exercise: it is a simple extension of the arithmetico-geometrical inequality.

4. **Lemma (3). ('Special case' of Theorem (1): 'for two positive numbers'.)**
Let u, v be positive real numbers. The inequality $\dfrac{u+v}{2} \geq \sqrt{uv}$ *holds. Equality holds iff $u = v$.*
Proof. Exercise.

5. **Illustration of the key idea in the proof of Theorem (1).**
   (a) We prove the statement ($\sharp$) below, which is the 'inequality part' of the 'special case' of Theorem (1) 'for four positive numbers':
   ($\sharp$) *Let a, b, c, d be positive real numbers.* $\dfrac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$.
   Proof of the statement ($\sharp$). Let a, b, c, d be positive real numbers.
   $\sqrt{a}, \sqrt{b}, \sqrt{ab}$ are well-defined, and $a = (\sqrt{a})^2$, $b = (\sqrt{b})^2$, $\sqrt{ab} = \sqrt{a}\sqrt{b}$.
   Then $a + b = (\sqrt{a})^2 + (\sqrt{b})^2 \geq 2\sqrt{a}\sqrt{b} = 2\sqrt{ab}$. Similarly, $c + d \geq 2\sqrt{cd}$.
   Therefore, (once again applying the same argument,) we have
$$\frac{a+b+c+d}{4} = \frac{1}{2}\left(\frac{a+b}{2} + \frac{c+d}{2}\right) \geq \frac{\sqrt{ab} + \sqrt{cd}}{2} \geq \sqrt[4]{ab}\sqrt[4]{cd} = \sqrt[4]{abcd}.$$

   (b) With the help of the statement ($\sharp$), we deduce the statement ($\flat$) below, which is the 'inequality part' of the 'special case' of Theorem (1) 'for three positive numbers':
   ($\flat$) *Let r, s, t be positive real numbers.* $\dfrac{r+s+t}{3} \geq \sqrt[3]{rst}$.
   Proof of the statement ($\flat$). Let r, s, t be positive real numbers.
   Define $u = \dfrac{r+s+t}{3}$. u is also a positive real number. By ($\sharp$), we have $\dfrac{r+s+t+u}{4} \geq \sqrt[4]{rstu}$.
   Note that $\dfrac{r+s+t+u}{4} = \dfrac{r+s+t+(r+s+t)/3}{4} = \dfrac{r+s+t}{3} = u$.
   Then $u = \dfrac{r+s+t+u}{4} \geq \sqrt[4]{rstu} = \sqrt[4]{rst} \cdot \sqrt[4]{u}$.
   Note that $u > 0$, and $\sqrt[4]{u} > 0$. Then $\left(\sqrt[4]{u}\right)^3 \geq \sqrt[4]{rst}$.
   Therefore $\dfrac{r+s+t}{3} = u = \left[\sqrt[3]{\left(\sqrt[4]{u}\right)^3}\right]^4 \geq \left(\sqrt[3]{\sqrt[4]{rst}}\right)^4 = \left(\sqrt[12]{rst}\right)^4 = \sqrt[3]{rst}$.

6. **Lemma (4). (Many 'special cases' of Theorem (1): 'for 2^n positive numbers'.)**
Let $n \in \mathbb{N}$. Let $a_1, a_2, \cdots, a_{2^n}$ be 2^n positive real numbers.
The inequality $\dfrac{a_1 + a_2 + \cdots + a_{2^n}}{2^n} \geq \sqrt[2^n]{a_1 a_2 \cdot \ldots \cdot a_{2^n}}$ *holds.*
Equality holds iff $a_1 = a_2 = \cdots = a_{2^n}$.

Proof of Lemma (4). Denote by $P(n)$ the proposition below:

'Suppose $a_1, a_2, \cdots, a_{2^n}$ are positive real numbers. Then

$$(a_1 a_2 \cdot \ldots \cdot a_{2^n})^{\frac{1}{2^n}} \leq \frac{a_1 + a_2 + \cdots + a_{2^n}}{2^n}.$$

Equality holds iff $a_1 = a_2 = \cdots = a_{2^n}$.'

- $P(0)$ is a trivially true statement. (Why?)
- Let $k \in \mathbb{N}$. Suppose $P(k)$ is true. Then the statement below is true:

'Suppose $c_1, c_2, \cdots, c_{2^k}$ are positive real numbers. Then

$$(c_1 c_2 \cdot \ldots \cdot c_{2^k})^{\frac{1}{2^k}} \leq \frac{c_1 + c_2 + \cdots + c_{2^k}}{2^k}.$$

Equality holds iff $c_1 = c_2 = \cdots = c_{2^k}$.'

We are going to verify that $P(k+1)$ is true:

Suppose $b_1, b_2, \cdots, b_{2^k}, \cdots, b_{2^{k+1}}$ are positive real numbers.

Write $d_1 = (b_1 b_2 \cdot \ldots \cdot b_{2^k})^{\frac{1}{2^k}}$ and $d_2 = (b_{2^k+1} b_{2^k+2} \cdot \ldots \cdot b_{2^{k+1}})^{\frac{1}{2^k}}$. Then

$$\begin{aligned}
(b_1 b_2 \cdot \ldots \cdot b_{2^{k+1}})^{\frac{1}{2^{k+1}}} &= \left[(b_1 b_2 \cdot \ldots \cdot b_{2^k})^{\frac{1}{2^k}} (b_{2^k+1} b_{2^k+2} \cdot \ldots \cdot b_{2^{k+1}})^{\frac{1}{2^k}} \right]^{\frac{1}{2}} = \sqrt{d_1 d_2} \\
&\leq \frac{d_1 + d_2}{2} \quad \text{(by Lemma (2))} \\
&= \frac{1}{2} \left[(b_1 b_2 \cdot \ldots \cdot b_{2^k})^{\frac{1}{2^k}} + (b_{2^k+1} b_{2^k+2} \cdot \ldots \cdot b_{2^{k+1}})^{\frac{1}{2^k}} \right] \\
&\leq \frac{1}{2} \left(\frac{b_1 + b_2 + \cdots + b_{2^k}}{2^k} + \frac{b_{2^k+1} + b_{2^k+2} + \cdots + b_{2^{k+1}}}{2^k} \right) \\
&= \frac{b_1 + b_2 + \cdots + b_{2^{k+1}}}{2^{k+1}}
\end{aligned}$$

* Suppose $b_1 = b_2 = \cdots = b_{2^{k+1}}$. Then

$$(b_1 b_2 \cdot \ldots \cdot b_{2^{k+1}})^{\frac{1}{2^{k+1}}} = b_1 = \frac{b_1 + b_2 + \cdots + b_{2^{k+1}}}{2^{k+1}}.$$

* Suppose

$$(b_1 b_2 \cdot \ldots \cdot b_{2^{k+1}})^{\frac{1}{2^{k+1}}} = \frac{b_1 + b_2 + \cdots + b_{2^{k+1}}}{2^{k+1}}.$$

Then

$$\left\{ \begin{aligned}
\frac{d_1 + d_2}{2} &= \sqrt{d_1 d_2} \\
\frac{b_1 + b_2 + \cdots + b_{2^k}}{2^k} &= (b_1 b_2 \cdot \ldots \cdot b_{2^k})^{\frac{1}{2^k}} \\
\frac{b_{2^k+1} + b_{2^k+2} + \cdots + b_{2^{k+1}}}{2^k} &= (b_{2^k+1} b_{2^k+2} \cdot \ldots \cdot b_{2^{k+1}})^{\frac{1}{2^k}}
\end{aligned} \right.$$

By the second and third equalities, we have $b_1 = b_2 = \cdots = b_{2^k}$ and $b_{2^k+1} = b_{2^k+2} = \cdots = b_{2^{k+1}}$ respectively. Then $d_1 = b_1$ and $d_2 = b_{2^k+1}$.

Since $\dfrac{d_1 + d_2}{2} = \sqrt{d_1 d_2}$, we have $b_1 = d_1 = d_2 = b_{2^k+1}$. Therefore $b_1 = b_2 = \cdots = b_{2^{k+1}}$.

Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

7. **Lemma (5). ('Backward Induction' Lemma.)**

Denote by $Q(m)$ the proposition below:

- *Suppose $a_1, a_2, \cdots, a_m$ are positive real numbers.*
  Then $(a_1 a_2 \cdot \ldots \cdot a_m)^{\frac{1}{m}} \leq \dfrac{a_1 + a_2 + \cdots + a_m}{m}$.
  Equality holds iff $a_1 = a_2 = \cdots = a_m$.

Let $p \in \mathbb{N}\backslash\{0\}$. Suppose $Q(p+1)$ is true. Then $Q(p)$ is true.

Proof of Lemma (5). Denote by $Q(m)$ the proposition below:

- Suppose $a_1, a_2, \cdots, a_m$ are positive real numbers.
  Then $(a_1 a_2 \cdot \ldots \cdot a_m)^{\frac{1}{m}} \leq \dfrac{a_1 + a_2 + \cdots + a_m}{m}$.
  Equality holds iff $a_1 = a_2 = \cdots = a_m$.

Let $p \in \mathbb{N}\backslash\{0\}$. The statements $Q(p+1)$, $Q(p)$ respectively read:

$Q(p+1)$: Suppose $c_1, c_2, \cdots, c_{p+1}$ are positive real numbers. Then $(c_1 c_2 \cdot \ldots \cdot c_{p+1})^{\frac{1}{p+1}} \leq \dfrac{c_1 + c_2 + \cdots + c_{p+1}}{p+1}$. Equality holds iff $c_1 = c_2 = \cdots = c_{p+1}$.

$Q(p)$: Suppose $b_1, b_2, \cdots, b_p$ are positive real numbers. Then $(b_1 b_2 \cdot \ldots \cdot b_p)^{\frac{1}{p}} \leq \dfrac{b_1 + b_2 + \cdots + b_p}{p}$. Equality holds iff $b_1 = b_2 = \cdots = b_p$.

Suppose the statement $Q(p+1)$ holds. We proceed to deduce that the statement $Q(p)$ holds:

$*$ Suppose $b_1, b_2, \cdots, b_p$ be positive real numbers.
Define $b_{p+1} = \dfrac{b_1 + b_2 + \cdots + b_p}{p}$. By definition, b_{p+1} is a positive real number. Then by $Q(p+1)$, we have

$$\frac{b_1 + b_2 + \cdots + b_p + b_{p+1}}{p+1} \geq (b_1 b_2 \cdot \ldots \cdot b_p \cdot b_{p+1})^{\frac{1}{p+1}}.$$

Note that

$$\frac{b_1 + b_2 + \cdots + b_p + b_{p+1}}{p+1} = \frac{b_1 + b_2 + \cdots + b_p + (b_1 + b_2 + \cdots + b_p)/p}{p+1} = \frac{b_1 + b_2 + \cdots + b_p}{p} = b_{p+1}.$$

Then

$$b_{p+1} = \frac{b_1 + b_2 + \cdots + b_p + b_{p+1}}{p+1} \geq (b_1 b_2 \cdot \ldots \cdot b_p \cdot b_{p+1})^{\frac{1}{p+1}} = (b_1 b_2 \cdot \ldots \cdot b_p)^{\frac{1}{p+1}} \cdot {b_{p+1}}^{\frac{1}{p+1}}$$

Note that ${b_{p+1}}^{\frac{1}{p+1}} > 0$. Then

$${b_{p+1}}^{\frac{p}{p+1}} = {b_{p+1}}^{1-\frac{1}{p+1}} \geq (b_1 b_2 \cdot \ldots \cdot b_p)^{\frac{1}{p+1}}.$$

Therefore

$$\frac{b_1 + b_2 + \cdots + b_p}{p} = b_{p+1} = {b_{p+1}}^{\frac{p}{p+1} \cdot \frac{p+1}{p}} \geq (b_1 b_2 \cdot \ldots \cdot b_p)^{\frac{1}{p+1} \cdot \frac{p+1}{p}} = (b_1 b_2 \cdot \ldots \cdot b_p)^{\frac{1}{p}}$$

$*$ Suppose $b_1 = b_2 = \cdots = b_p$.
Then

$$(b_1 b_2 \cdot \ldots \cdot b_p)^{\frac{1}{p}} = b_1 = \frac{b_1 + b_2 + \cdots + b_p}{p}.$$

$*$ Suppose

$$(b_1 b_2 \cdot \ldots \cdot b_p)^{\frac{1}{p}} = \frac{b_1 + b_2 + \cdots + b_p}{p}.$$

Then $(b_1 b_2 \cdot \ldots \cdot b_p)^{\frac{1}{p}} = b_{p+1}$ by the definition of b_{p+1}.
Therefore

$$(b_1 b_2 \cdot \ldots \cdot b_p \cdot b_{p+1})^{\frac{1}{p+1}} = \left({b_{p+1}}^{p+1}\right)^{\frac{1}{p+1}} = b_{p+1} = \frac{p b_{p+1} + b_{p+1}}{p+1} = \frac{b_1 + b_2 + \cdots + b_p + b_{p+1}}{p+1}$$

Therefore $b_1 = b_2 = \cdots = b_p = b_{p+1}$. In particular, $b_1 = b_2 = \cdots = b_p$.

Hence the statement $Q(p)$ holds.

8. **Lemma (6).**

   Let $k \in \mathbb{N} \backslash \{0, 1\}$. There exists some $h \in \mathbb{N}$ so that $2^h < k \leq 2^{h+1}$.

   Proof of Lemma (6).

   Let $k \in \mathbb{N} \backslash \{0, 1\}$. Note that $2^k \geq k$. (Why?)

   Define the set $S = \{x \in \mathbb{N} : 2^x \geq k\}$. [Apply the Well-ordering Principle for integers on the set S.]

   We have $k \in S$. Then S is a non-empty subset of $\mathbb{N}$. By the Well-ordering Principle for integers, S has a least element, which we denote by λ.

   Since $k \geq 2$, we have $\lambda \geq 1$.

   Define $h = \lambda - 1$. By definition, $h \in \mathbb{N}$ and $k \leq 2^{h+1}$.

   We verify that $2^h < k$:

   - Suppose it were true that $2^h \geq k$. Then $h \in S$. But $h < \lambda$. Contradiction arises.
     Hence $2^h < k$ in the first place.

   The result follows.

9. **Proof of Theorem (1).**

   Denote by $Q(m)$ the proposition below:

   - Suppose $a_1, a_2, \cdots, a_m$ are positive real numbers. Then $(a_1 a_2 \cdot \ldots \cdot a_m)^{\frac{1}{m}} \leq \dfrac{a_1 + a_2 + \cdots + a_m}{m}$. Equality holds iff $a_1 = a_2 = \cdots = a_m$.

   $Q(1)$ is (trivially) true.

   By Lemma (4), $Q(2^M)$ is true for any $M \in \mathbb{N}$.

   Let $k \in \mathbb{N} \backslash \{0, 1\}$. By Lemma (6), there exists some $h \in \mathbb{N}$ such that $2^h < k \leq 2^{h+1}$.

   By Lemma (5), since $Q(2^{h+1})$ is true, $Q(2^{h+1} - 1)$ is true as well. Then, repeatedly applying Lemma (5), we deduce that $Q(2^{h+1} - 2)$ is true, $Q(2^{h+1} - 3)$ is true, ..., $Q(k + 1)$ is true, and $Q(k)$ is true.

   It follows that $Q(m)$ is true for any $m \in \mathbb{N} \backslash \{0\}$.

10. **'Backward induction' method.**

    The argument above for the Arithmetico-geometrical Inequality is an example of 'backward induction'.

    Recall this convention on notation:

    - *Let $N \in \mathbb{Z}$. $[\![N, +\infty)$ stands for the set $\{x \in \mathbb{Z} : x \geq N\}$.*

    Theorem (7). ('Principle of "Backward induction"'.)

    Let $Q(n)$ be a predicate with variable n. Let $\{A_n\}_{n=0}^{\infty}$ be a strictly increasing sequence of integers.

    Suppose that all of (†), (‡), ($\star$) are true:

    (†) The statement $Q(A_0)$ is true.

    (‡) For any $k \in \mathbb{N}$, if the statement $Q(A_k)$ is true then the statement $Q(A_{k+1})$ is true.

    ($\star$) For any $m \in [\![A_0, +\infty)$, if the statement $Q(m)$ is true then the statement $Q(m - 1)$ is true.

    Then the statement $Q(n)$ is true for any $n \in [\![A_0, +\infty)$.

    Theorem (8). (Set-theoretic formulation of 'Principle of "Backward induction"'.)

    Let T be a subset of $[\![A_0, +\infty)$. Let $\{A_n\}_{n=0}^{\infty}$ be a strictly increasing sequence of integers.

    Suppose that all of (†), (‡), ($\star$) are true:

    (†) $A_0 \in T$.

    (‡) For any $k \in \mathbb{N}$, if $A_k \in T$ then $A_{k+1} \in T$.

    ($\star$) For any $m \in [\![A_0, +\infty)$, if $m \in T$ then $m - 1 \in T$.

    Then $T = [\![A_0, +\infty)$.

    The proofs of Theorem (7), Theorem (8) are left as exercises. As statements, Theorem (7) and Theorem (8) are logically equivalent. Theorem (7) suggests a scheme in its application; write down the scheme as an exercise. (A concrete example on how the scheme works is illustrated by the argument in Lemma (4) and Lemma (5).)