1. Definitions. (Arithmetic mean, geometric mean and harmonic mean.)

Let $n \in \mathbb{N} \backslash \{0\}$. Let $a_1, a_2, \cdots, a_n$ be n positive real numbers.

(a) *The number*

$$\frac{a_1 + a_2 + \cdots + a_n}{n}$$

is called the **arithmetic mean** *of $a_1, a_2, \cdots, a_n$.*

(b) *The number*

$$\sqrt[n]{a_1 a_2 \cdot \ldots \cdot a_n}$$

is called the **geometric mean** *of $a_1, a_2, \cdots, a_n$.*

(c) *The number*

$$\left[\frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)\right]^{-1}$$

is called the **harmonic mean** *of $a_1, a_2, \cdots, a_n$.*

Remark. By definition, the harmonic mean of $a_1, a_2, \cdots, a_n$ is the reciprocal of the arithmetic mean of the reciprocals of $a_1, a_2, \cdots, a_n$.

2. **Theorem (1). (Arithmetico-geometrical Inequality.)**

Let $m \in \mathbb{N} \backslash \{0\}$. Let $a_1, a_2, \cdots, a_m$ be m posiitve real numbers.

The inequality

$$\frac{a_1 + a_2 + \cdots + a_m}{m} \geq \sqrt[m]{a_1 a_2 \cdot \ldots \cdot a_m}$$

holds. Equality holds iff $a_1 = a_2 = \cdots = a_m$.

3. **Corollary (2).**

Let $m \in \mathbb{N} \backslash \{0\}$. Let $a_1, a_2, \cdots, a_m$ be m positive real numbers.

The inequality

$$\left[\frac{1}{m}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_m}\right)\right]^{-1} \leq \sqrt[m]{a_1 a_2 \cdot \ldots \cdot a_m} \leq \frac{a_1 + a_2 + \cdots + a_m}{m}$$

holds. Each equality holds iff $a_1 = a_2 = \cdots = a_m$.

4. **Lemma (3). ('Special case' of Theorem (1): 'for two positive numbers'.)**

Let u, v be positive real numbers.

The inequality $\dfrac{u+v}{2} \geq \sqrt{uv}$ holds. Equality holds iff $u = v$.

5. **Illustration of the key idea in the proof of Theorem (1).**

(a) We prove the statement ($\sharp$) below, which is the 'inequality part' of the 'special case' of Theorem (1) 'for four positive numbers':

($\sharp$) *Let a, b, c, d be positive real numbers.* $\dfrac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$.

Proof of the statement ($\sharp$).

Let a, b, c, d be positive real numbers.

This is actually the argument for the 'inequality part' of Lemma (3).

$\sqrt{a}, \sqrt{b}, \sqrt{ab}$ are well-defined, and $a = (\sqrt{a})^2$, $b = (\sqrt{b})^2$, $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

Then $a+b = (\sqrt{a})^2 + (\sqrt{b})^2 \geq 2\sqrt{a} \cdot \sqrt{b} = 2\sqrt{ab}$.

Therefore $\dfrac{a+b}{2} \geq \sqrt{ab}$.

Similarly, we have $\dfrac{c+d}{2} \geq \sqrt{cd}$.

Now, once again applying the same argument above, we have

$$\frac{a+b+c+d}{4} = \frac{1}{2}\left(\frac{a+b}{2} + \frac{c+d}{2}\right)$$

$$\geq \frac{1}{2}\left(\sqrt{ab} + \sqrt{cd}\right) \geq \sqrt{\sqrt{ab} \cdot \sqrt{cd}} = \sqrt[4]{abcd}. \quad \square$$

Illustration of the key idea in the proof of Theorem (1).

(a) We prove the statement (♯) below:

(♯) *Let a, b, c, d be positive real numbers.* $\dfrac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$.

(b) With the help of the statement (♯), we deduce the statement (♭) below, which is the 'inequality part' of the 'special case' of Theorem (1) 'for three positive numbers':

(♭) *Let r, s, t be positive real numbers.* $\dfrac{r+s+t}{3} \geq \sqrt[3]{rst}$.

Proof of the statement (♭).

Let r, s, t be positive real numbers.

Define $u = \dfrac{r+s+t}{3}$. Note that u is also a positive real number.

By (♯), we have $\dfrac{r+s+t+u}{4} \geq \sqrt[4]{rstu}$.

Note that $\dfrac{r+s+t+u}{4} = \dfrac{r+s+t+(r+s+t)/3}{4} = \dfrac{r+s+t}{3} = u$.

Then $u = \dfrac{r+s+t+u}{4} \geq \sqrt[4]{rstu} = \sqrt[4]{rst} \cdot \sqrt[4]{u}$.

Note that $\sqrt[4]{u} > 0$. Then $(\sqrt[4]{u})^3 = u / \sqrt[4]{u} \geq \sqrt[4]{rst}$.

Therefore $\dfrac{r+s+t}{3} = u = \left[(\sqrt[4]{u})^3\right]^{\frac{4}{3}} \geq (\sqrt[4]{rst})^{\frac{4}{3}} = \sqrt[3]{rst}$. □

6. **Lemma (4). (Many 'special cases' of Theorem (1): 'for 2^n positive numbers'.)**

Let $n \in \mathbb{N}$. Let $a_1, a_2, \cdots, a_{2^n}$ be 2^n positive real numbers.

The inequality $\dfrac{a_1 + a_2 + \cdots + a_{2^n}}{2^n} \geq \sqrt[2^n]{a_1 a_2 \cdot \ldots \cdot a_{2^n}}$ *holds.*

Equality holds iff $a_1 = a_2 = \cdots = a_{2^n}$.

Proof? Apply mathematical induction.

7. **Lemma (5). ('Backward Induction' Lemma.)**

Denote by $Q(m)$ the proposition below:

- *Suppose $a_1, a_2, \cdots, a_m$ are positive real numbers.*

  Then $(a_1 a_2 \cdot \ldots \cdot a_m)^{\frac{1}{m}} \leq \dfrac{a_1 + a_2 + \cdots + a_m}{m}$.

  Equality holds iff $a_1 = a_2 = \cdots = a_m$.

Let $p \in \mathbb{N} \backslash \{0\}$. Suppose $Q(p+1)$ is true. Then $Q(p)$ is true.

Proof? Imitate the argument for deducing $Q(3)$ from $Q(4)$.

8. **Lemma (6).**

Let $k \in \mathbb{N} \backslash \{0, 1\}$. There exists some $h \in \mathbb{N}$ so that $2^h < k \leq 2^{h+1}$.

Proof? Apply the Well-ordering Principle for Integers.

9. **Proof of Theorem (1).**

Denote by $Q(m)$ the proposition below:

- Suppose $a_1, a_2, \cdots, a_m$ are positive real numbers. Then $(a_1 a_2 \cdot \ldots \cdot a_m)^{\frac{1}{m}} \leq \dfrac{a_1 + a_2 + \cdots + a_m}{m}$. Equality holds iff $a_1 = a_2 = \cdots = a_m$.

$Q(1)$ is (trivially) true. (Why?)

By Lemma (4), $Q(2^M)$ is true for any $M \in \mathbb{N}$.

Pick any $k \in \mathbb{N} \setminus \{0, 1\}$. [Ask: Is $Q(k)$ true?]

By Lemma (6), there exists some $h \in \mathbb{N}$ such that $2^h < k \leq 2^{h+1}$.

By Lemma (5), since $Q(2^{h+1})$ is true, $Q(2^{h+1} - 1)$ is also true.

Then, repeatedly applying Lemma (5), we deduce in succession that $Q(2^{h+1} - 2)$ is true, $Q(2^{h+1} - 3)$ is true, ..., $Q(k+1)$ is true, and $Q(k)$ is true. [This is a repeated application of Modus Ponens.]

It follows that $Q(m)$ is true for any $m \in \mathbb{N} \setminus \{0\}$.

10. **'Backward induction' method.**

The argument above for the Arithmetico-geometrical Inequality is an example of 'backward induction'.

Recall this convention on notation:

- *Let $N \in \mathbb{Z}$. $[\![N, +\infty)$ stands for the set $\{x \in \mathbb{Z} : x \geq N\}$.*

Theorem (7). ('Principle of "Backward induction"'.)

Let $Q(n)$ be a predicate with variable n. Let $\{A_n\}_{n=0}^{\infty}$ be a strictly increasing sequence of integers.

Suppose that all of $(\dagger)$, $(\ddagger)$, $(\star)$ are true:

($\dagger$) The statement $Q(A_0)$ is true.

($\ddagger$) For any $k \in \mathbb{N}$, if the statement $Q(A_k)$ is true then the statement $Q(A_{k+1})$ is true.

($\star$) For any $m \in [\![A_0, +\infty)$, if the statement $Q(m)$ is true then the statement $Q(m-1)$ is true.

Then the statement $Q(n)$ is true for any $n \in [\![A_0, +\infty)$.

Theorem (8). (Set-theoretic formulation of 'Principle of "Backward induction"'.)
Let T be a subset of $[\![A_0, +\infty)$. Let $\{A_n\}_{n=0}^{\infty}$ be a strictly increasing sequence of integers.

Suppose that all of (†), (‡), ($\star$) are true:

(†) $A_0 \in T$.

(‡) For any $k \in \mathbb{N}$, if $A_k \in T$ then $A_{k+1} \in T$.

($\star$) For any $m \in [\![A_0, +\infty)$, if $m \in T$ then $m - 1 \in T$.

Then $T = [\![A_0, +\infty)$.

The proofs of Theorem (7), Theorem (8) are left as exercises.

As statements, Theorem (7) and Theorem (8) are logically equivalent.

Theorem (7) suggests a scheme in its application; write down the scheme as an exercise.

(A concrete example on how the scheme works is illustrated by the argument in Lemma (4) and Lemma (5).)