MATH1050 Monotonicity and boundedness for infinite sequences of real numbers

1. **Definition.**
Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{R}$.

(a) Let $\kappa \in \mathbb{R}$. *κ is said to be* $\left\{ \begin{array}{c} \textbf{upper bound} \\ \textbf{lower bound} \end{array} \right\}$ **of** $\{a_n\}_{n=0}^{\infty}$ **in** $\mathbb{R}$ *if, for any $n \in \mathbb{N}$,* $\left\{ \begin{array}{c} a_n \leq \kappa \\ a_n \geq \kappa \end{array} \right\}$.

(b) $\{a_n\}_{n=0}^{\infty}$ *is said to be* $\left\{ \begin{array}{c} \textbf{bounded above} \\ \textbf{bounded below} \end{array} \right\}$ **in** $\mathbb{R}$ *if there exists some $\kappa \in \mathbb{R}$ such that for any $n \in \mathbb{N}$,* $\left\{ \begin{array}{c} a_n \leq \kappa \\ a_n \geq \kappa \end{array} \right\}$.

Bounded-ness for infinite sequences of real numbers can be re-formulated in terms of bounded-ness for their corresponding 'sets of all terms'.

Lemma (1).
Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{R}$. Define $T(\{a_n\}_{n=0}^{\infty}) = \{x \in \mathbb{R} : x = a_n$ for some $n \in \mathbb{N}\}$.
($T(\{a_n\}_{n=0}^{\infty})$ is the set of all terms of $\{a_n\}_{n=0}^{\infty}$.)
The statements below hold:

(a) *$\{a_n\}_{n=0}^{\infty}$ is bounded above in $\mathbb{R}$ by β iff $T(\{a_n\}_{n=0}^{\infty})$ is bounded above in $\mathbb{R}$ by β.*
(b) *$\{a_n\}_{n=0}^{\infty}$ is bounded below in $\mathbb{R}$ by β iff $T(\{a_n\}_{n=0}^{\infty})$ is bounded below in $\mathbb{R}$ by β.*

Proof of (a). Exercise. (Word game.)

2. **Definition.**
Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{R}$.

(a) $\{a_n\}_{n=0}^{\infty}$ *is said to be* $\left\{ \begin{array}{c} \textbf{increasing} \\ \textbf{decreasing} \end{array} \right\}$ *if, for any $n \in \mathbb{N}$,* $\left\{ \begin{array}{c} a_n \leq a_{n+1} \\ a_n \geq a_{n+1} \end{array} \right\}$.

(b) $\{a_n\}_{n=0}^{\infty}$ *is said to be* $\left\{ \begin{array}{c} \textbf{strictly increasing} \\ \textbf{strictly decreasing} \end{array} \right\}$ *if, for any $n \in \mathbb{N}$,* $\left\{ \begin{array}{c} a_n < a_{n+1} \\ a_n > a_{n+1} \end{array} \right\}$.

Remarks on terminology.

(a) $\{a_n\}_{n=0}^{\infty}$ is said to be **monotonic** if $\{a_n\}_{n=0}^{\infty}$ is increasing or decreasing.
(b) $\{a_n\}_{n=0}^{\infty}$ is said to be **strictly monotonic** if $\{a_n\}_{n=0}^{\infty}$ is strictly increasing or strictly decreasing.

Lemma (2).
Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{R}$. Define $T(\{a_n\}_{n=0}^{\infty}) = \{x \in \mathbb{R} : x = a_n$ for some $n \in \mathbb{N}\}$. ($T(\{a_n\}_{n=0}^{\infty})$ is the set of all terms of $\{a_n\}_{n=0}^{\infty}$.) The statements below hold:

(a) *Suppose $\{a_n\}_{n=0}^{\infty}$ is strictly increasing. Then $T(\{a_n\}_{n=0}^{\infty})$ has no greatest element.*
(b) *Suppose $\{a_n\}_{n=0}^{\infty}$ is strictly decreasing. Then $T(\{a_n\}_{n=0}^{\infty})$ has no least element.*

Proof. [We give an argument for (a) only; that for (b) is similar.]
Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{R}$. Define $T(\{a_n\}_{n=0}^{\infty}) = \{x \in \mathbb{R} : x = a_n$ for some $n \in \mathbb{N}\}$.
Suppose $\{a_n\}_{n=0}^{\infty}$ is strictly increasing.
Suppose it were true that $T(\{a_n\}_{n=0}^{\infty})$ had a greatest element, say, λ.
By definition, $\lambda \in T(\{a_n\}_{n=0}^{\infty})$.
[Ask: Is there any element of $T(\{a_n\}_{n=0}^{\infty})$ greater than λ. How to conceive it, if there is any?]
By definition, there existed some $n_0 \in \mathbb{N}$ such that $\lambda = a_{n_0}$.
Note that $n_0 + 1 \in \mathbb{N}$.
Define $x = a_{n_0+1}$. By definition, $x \in T(\{a_n\}_{n=0}^{\infty})$.
Since $\{a_n\}_{n=0}^{\infty}$ is strictly increasing, we would have $x = a_{n_0+1} > a_{n_0} = \lambda$.
But λ was a greatest element of $T(\{a_n\}_{n=0}^{\infty})$. Contradition arises.
It follows that in the first place, $T(\{a_n\}_{n=0}^{\infty})$ has no greatest element.

3. **Example (1).**

For any $n \in \mathbb{N}$, define $a_n = \dfrac{(n+1)(n+4)}{(n+2)(n+3)}$. Define $T = \{x \mid x = a_n \text{ for some } n \in \mathbb{N}\}$.

(a) *$\{a_n\}_{n=0}^{\infty}$ is bounded above in $\mathbb{R}$ by 1. (Equivalently, T is bounded above in $\mathbb{R}$ by 1.)*

Proof.

Let $n \in \mathbb{N}$. We have $a_n = \dfrac{(n+1)(n+4)}{(n+2)(n+3)} = \dfrac{n^2+5n+4}{n^2+5n+6} = 1 - \dfrac{2}{n^2+5n+6} \leq 1 - 0 = 1$.

Hence $\{a_n\}_{n=0}^{\infty}$ is bounded above by 1.

(b) *$\{a_n\}_{n=0}^{\infty}$ is strictly increasing.*

Proof.

Let $n \in \mathbb{N}$.

$$\begin{aligned} a_{n+1} - a_n &= \frac{[(n+1)+1][(n+1)+4]}{[(n+1)+2][(n+1)+3]} - \frac{(n+1)(n+4)}{(n+2)(n+3)} = \frac{(n+2)(n+5)}{(n+3)(n+4)} - \frac{(n+1)(n+4)}{(n+2)(n+3)} \\ &= \frac{(n+2)^2(n+5)-(n+1)(n+4)^2}{(n+2)(n+3)(n+4)} = \frac{4}{(n+2)(n+3)(n+4)} > 0 \end{aligned}$$

Then $a_{n+1} > a_n$.

(c) *T has no greatest element.*

Proof.

Suppose T had a greatest element, say, λ. Then there would exist some $n \in \mathbb{N}$ such that $\lambda = a_n$.

Take $x_0 = a_{n+1}$. By definition, $x_0 \in T$.

We have $x_0 = a_{n+1} > a_n = \lambda$. Therefore λ would not be a greatest element of T. Contradiction arises.

It follows that T has no greatest element in the first place.

(d) *For any $\beta \in \mathbb{R}$, if $\beta \geq 1$ then β is an upper bound of T in $\mathbb{R}$.* (Exercise.)

(e) *For any $\beta \in \mathbb{R}$, if $\beta < 1$ then β is not an upper bound of T in $\mathbb{R}$.*

Remark. We assume the validity of the statement (AP) below (which is known as the Archimedean Principle):

(AP) *For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.*

Proof.

Pick any $\beta \in \mathbb{R}$. Suppose $\beta < 1$.

[We apply the proof-by-contradiction method to prove that β is not an upper bound of T in $\mathbb{R}$.]

- Suppose β were an upper bound of T in $\mathbb{R}$.

  [Ask: Is there any element of T greater than β?
  We may reformulate the question in this way: can we name an appropriate natural number k for which $a_k > \beta$ holds?
  This suggests we study the inequality '$a_m > \beta$'. We try to re-formulate it into the form '$P(m)Q(\beta) > 1$', in which $P(m)$ is an expression for some integer depending on m but not β, and $Q(\beta)$ is an expression for some positive real number depending on β but not m.
  This provides a hint on how we may apply Statement (AP) to name an appropriate m for which $a_m > \beta$ holds.]

  Define $\varepsilon = \dfrac{1-\beta}{2}$. By definition, $\varepsilon > 0$.

  By (AP), since $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.

  For the same ε, N, we have $(N^2+5N+6)\varepsilon \geq N\varepsilon > 1$.

  Then $\dfrac{2}{N^2+5N+6} < 2\varepsilon = 1-\beta$. Therefore $\beta < 1 - \dfrac{2}{N^2+5N+6} = a_N$.

  But β was an upper bound of T in $\mathbb{R}$. Contradiction arises.

(f) *T has a supremum in $\mathbb{R}$, namely 1.*

Proof.

The set of all upper bounds of T in $\mathbb{R}$ is the interval $[1, +\infty)$, whose least element is 1.

4. **Example (2).**

For any $n \in \mathbb{N}$, define $a_n = \displaystyle\sum_{k=0}^{n} \frac{9}{10^{k+1}}$. Define $T = \{x \mid x = a_n \text{ for some } n \in \mathbb{N}\}$.

(a) *$\{a_n\}_{n=0}^{\infty}$ is bounded above in $\mathbb{R}$ by 1. (Equivalently, T is bounded above in $\mathbb{R}$ by 1.)*

Proof.

Let $n \in \mathbb{N}$. We have $a_n = \sum_{k=0}^{n} \frac{9}{10^{k+1}} = \frac{9}{10} \cdot \frac{1 - 1/10^{n+1}}{1 - 1/10} = 1 - \frac{1}{10^{n+1}} \leq 1 - 0 = 1$.

Hence $\{a_n\}_{n=0}^{\infty}$ is bounded above by 1.

(b) *$\{a_n\}_{n=0}^{\infty}$ is strictly increasing.*

Proof.

Let $n \in \mathbb{N}$. We have $a_{n+1} - a_n = \sum_{k=0}^{n+1} \frac{9}{10^{k+1}} - \sum_{k=0}^{n} \frac{9}{10^{k+1}} = \frac{9}{10^{n+2}} > 0$. Then $a_{n+1} > a_n$.

(c) *T has no greatest element.* (Exercise.)

(d) *For any $\beta \in \mathbb{R}$, if $\beta \geq 1$ then β is an upper bound of T in $\mathbb{R}$.* (Exercise.)

(e) *For any $\beta \in \mathbb{R}$, if $\beta < 1$ then β is not an upper bound of T in $\mathbb{R}$.*

Remark. We assume the validity of the statement (AP) below (which is known as the Archimedean Principle):

(AP) *For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.*

Proof.

Pick any $\beta \in \mathbb{R}$. Suppose $\beta < 1$.

[We apply the proof-by-contradiction method to prove that β is not an upper bound of T in $\mathbb{R}$.]

- Suppose β were an upper bound of T in $\mathbb{R}$.

  [Ask: Is there any element of T greater than β?
  We may reformulate the question in this way: can we name an appropriate natural number k for which $a_k > \beta$ holds?
  This suggests we study the inequality '$a_m > \beta$'. We try to re-formulate it into the form '$P(m)Q(\beta) > 1$', in which $P(m)$ is an expression for some integer depending on m but not β, and $Q(\beta)$ is an expression for some positive real number depending on β but not m.
  This provides a hint on how we may apply Statement (AP) to name an appropriate m for which $a_m > \beta$ holds.]

  Define $\varepsilon = 1 - \beta$. By definition, $\varepsilon > 0$.
  By (AP), since $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.
  For the same ε, N, we have $10^{N+1}\varepsilon \geq N\varepsilon > 1$.

  Then $\frac{1}{10^{N+1}} < \varepsilon = 1 - \beta$. Therefore $\beta < 1 - \frac{1}{10^{N+1}} = \sum_{k=0}^{N} \frac{9}{10^{k+1}} = a_N$.

  But β was an upper bound of T in $\mathbb{R}$. Contradiction arises.

(f) *T has a supremum in $\mathbb{R}$, namely 1.* (Exercise.)

5. **Example (3).**

Let p be a positive prime number. Define $\alpha = \sqrt{p}$. Let $b \in (\alpha, +\infty)$.

Let $\{a_n\}_{n=0}^{\infty}$ be the infinite sequence defined recursively by

$$\begin{cases} a_0 &= b \\ a_{n+1} &= \frac{1}{2}(a_n + \frac{\alpha^2}{a_n}) \quad \text{for any} \quad n \in \mathbb{N} \end{cases}$$

Define $T = \{x \in \mathbb{R} : x = a_n$ for some $n \in \mathbb{N}\}$.

The infinite sequence $\{a_n\}_{n=0}^{\infty}$ will provide approximations $a_0, a_1, a_2, a_3, \cdots$ as close to the irrational number $\alpha = \sqrt{p}$ as we like, as the index in the a_j's increases.

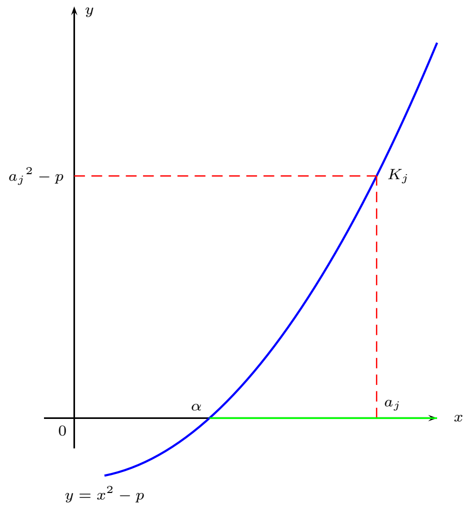
It is done as described below algorithmically:

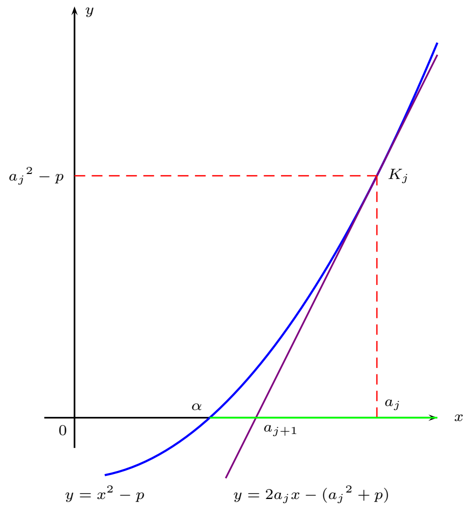
- Consider the curve $C : y = x^2 - p$ on the coordinate plane. (C intersects the x-axis at the point $(\sqrt{p}, 0)$.)

  Take $M_0 = (a_0, 0) = (b, 0)$.

  For each $j \in \mathbb{N}$, draw the line ℓ_j (whose equation is $x = a_j$) through M_j perpendicular to the x-axis. The intersection of ℓ_j with C is defined to be K_j: its coordinates are given by $K_j = (a_j, {a_j}^2 - p)$.

  Draw the tangent t_j to the curve C at K_j. (The equation of t_j is $y = 2a_jx - ({a_j}^2 + p)$.) The intersection of t_j with the x-axis is defined to be $M_{j+1} = (a_{j+1}, 0)$.





Here are the properties of the infinite sequence $\{a_n\}_{n=0}^{\infty}$ and the set T:

(a) *For any $n \in \mathbb{N}$, $a_n > \alpha$.*

Remark. As a consequence, $\{a_n\}_{n=0}^{\infty}$ is bounded below in $\mathbb{R}$ by α, and T is bounded below in $\mathbb{R}$ by α.

Proof.
Note that $a_0 = b > \alpha > 0$.

Suppose it were true that there existed some $n \in \mathbb{N}$ so that $a_n \leq \alpha$.
Then there would be a smallest $N \in \mathbb{N}$ so that $a_N > \alpha$ and $a_{N+1} \leq \alpha$. (Why? Apply the Well-ordering Principle for Integers.)

We would have $\alpha \geq a_{N+1} = \dfrac{1}{2}\left(a_N + \dfrac{\alpha^2}{a_N}\right) = \dfrac{a_N{}^2 + \alpha^2}{2a_N}$.

Since $a_N > 0$, we would have $2\alpha a_N \geq a_N{}^2 + \alpha^2$. Then $(a_N - \alpha)^2 = a_N{}^2 - 2\alpha a_N + \alpha^2 \leq 0$. Therefore $a_N = \alpha$. But $a_N > \alpha$.
Contradiction arises.

Hence, in the first place, we have $a_n > \alpha$ for any $n \in \mathbb{N}$.

(b) *$\{a_n\}_{n=0}^{\infty}$ is strictly decreasing.*

Proof.
Let $n \in \mathbb{N}$. $a_n > \alpha > 0$. Then $a_n{}^2 - \alpha^2 > 0$ also.

Therefore $a_{n+1} - a_n = \dfrac{1}{2}\left(a_n + \dfrac{\alpha^2}{a_n}\right) - a_n = \dfrac{1}{2}\left(-a_n + \dfrac{\alpha^2}{a_n}\right) = -\dfrac{a_n{}^2 - \alpha^2}{2a_n} < 0$.

Hence $a_{n+1} > a_n$.

(c) $0 < a_n - \alpha < \dfrac{b - \alpha}{2^n}$ *for any $n \in \mathbb{N}\backslash\{0\}$.*

Remark. Heuristically speaking, the infinite sequence $\{a_n\}_{n=0}^{\infty}$ will descend to as close as α as we like, but it will never 'reach' α.

Proof.
Let $n \in \mathbb{N}$. For each $k = 0, 1, 2, \cdots, n$, since $a_k > \alpha$, we have $a_k - \alpha > 0$. We have

$$\begin{aligned} a_n - \alpha &= \frac{1}{2}\left(a_{n-1} + \frac{\alpha^2}{a_{n-1}}\right) - \alpha = \frac{1}{2}(a_{n-1} - \alpha) + \frac{1}{2}\left(\frac{\alpha^2}{a_{n-1}} - \alpha\right) \\ &= \frac{1}{2}(a_{n-1} - \alpha) - \frac{\alpha}{2} \cdot \frac{a_{n-1} - \alpha}{a_{n-1}} < \frac{1}{2}(a_{n-1} - \alpha) < \frac{1}{2^2}(a_{n-2} - \alpha) < \cdots < \frac{1}{2^n}(a_0 - \alpha) = \frac{1}{2^n}(b - \alpha) \end{aligned}$$

(d) *T has no least element.*

Proof.
Suppose T had a least element, say, λ.
Then there would exist some $n_0 \in \mathbb{N}$ such that $\lambda = a_{n_0}$.
Now take $x_0 = a_{n_0+1}$. We would have $x_0 \in T$ and $x_0 = a_{n_0+1} < a_{n_0} = \lambda$.
Contradiction arises. Hence T has no least element in the first place.

(e) *For any $\beta \in \mathbb{R}$, if $\beta < \alpha$ then β is a lower bound of T in $\mathbb{R}$.* (Exercise.)

(f) *For any $\beta \in \mathbb{R}$, if $\beta > \alpha$ then β is not a lower bound of T in $\mathbb{R}$.*

Remark. We assume the validity of the statement (AP) below (which is known as the Archimedean Principle):

(AP) *For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.*

Proof.
Pick any $\beta \in \mathbb{R}$. Suppose $\beta > \alpha$.
[We apply the proof-by-contradiction method to prove that β is not a lower bound of T in $\mathbb{R}$.]

- Suppose β were a lower bound of T in $\mathbb{R}$.
  [Ask: Is there any element of T less than β?
  We may reformulate the question in this way: can we name an appropriate natural number k for which $a_k < \beta$ holds?
  This seems to be difficult to answer. So we further reformulate the question: can we name an appropriate natural number k for which $a_k - \alpha < \beta - \alpha$ holds?
  This further reformulation is helpful because we know for sure that $0 < a_m - \alpha < \dfrac{b-\alpha}{2^m}$ for each $m \in \mathbb{N}$.
  This suggests we ask for something 'more demanding': can we name an appropriate natural number k for which $a_k - \alpha < \dfrac{b-\alpha}{2^k} < \beta - \alpha$ holds?
  This provides a hint on how we may apply Statement (AP) to answer the original question.]
  Define $\varepsilon = \dfrac{\beta-\alpha}{b-\alpha}$. By definition, $\varepsilon > 0$.
  By (AP), since $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.
  For the same ε, N, we have $2^N\varepsilon \geq N\varepsilon > 1$.
  Then $2^N \cdot \dfrac{\beta-\alpha}{b-\alpha} > 1$. Therefore $a_{2^N} - \alpha < \dfrac{b-\alpha}{2^N} < \beta - \alpha$. Hence $a_{2^N} < \beta$.
  But β was a lower bound of T in $\mathbb{R}$. Contradiction arises.

(g) *T has an infimum in $\mathbb{R}$, namely, α.*
Proof.
The set of all lower bounds of T in $\mathbb{R}$ is the interval $(-\infty, \alpha]$, whose least element is α.

Further remarks.

(1) The idea and the calculation will still work even when we do not require p to be a positive prime number; we may allow p to be any positive real number that we like. The infinite sequence $\{a_n\}_{n=0}^{\infty}$ will provide approximations which descends to $\alpha = \sqrt{p}$ as close as we like, but never reaches α.

(2) How about finding cubic roots of positive real numbers?
Suppose p is a positive real number and $\alpha = \sqrt[3]{p}$. Suppose $b \in (\alpha, +\infty)$. Define infinite sequence $\{a_n\}_{n=0}^{\infty}$ recursively by
$$\begin{cases} a_0 &= b \\ a_{n+1} &= \dfrac{1}{3}(2a_n + \dfrac{\alpha^3}{{a_n}^2}) \quad \textit{for any} \quad n \in \mathbb{N} \end{cases}$$
This infinite sequence $\{a_n\}_{n=0}^{\infty}$ will provide approximations which descends to $\alpha = \sqrt[3]{p}$ as close as we like, but never reaches α.
(First draw the picture and formulate the algorithm which are analogous to the ones for the original example on square roots. This will give you a feeling on what this infinite sequence is 'doing'. Then try to formulate and prove some statements which are analogous to the ones that we have proved for the original example.)

(3) Can you generalize the idea to finding quartic roots of positive real numbers? Quintic roots? n-th roots?

(4) The idea and method described here is a 'concrete' example of the application of **Newton's Method (for finding approximate solutions of equations)**.

6. **Some 'coincidence' in Examples (1), (2), (3).**

We make some observations on Examples (1), (2), (3).

Example (1).

- *The infinite sequence $\left\{\dfrac{(n+1)(n+4)}{(n+2)(n+3)}\right\}_{n=0}^{\infty}$ is increasing and bounded above in $\mathbb{R}$.*
  The supremum of its set of all terms is 1. Coincidentally, the limit of this infinite sequence is also 1.

Example (2).

- *The infinite sequence $\left\{\displaystyle\sum_{k=0}^{n} \dfrac{9}{10^{k+1}}\right\}_{n=0}^{\infty}$ is increasing and bounded above in $\mathbb{R}$.*
  The supremum of its set of all terms is 1. Coincidentally, the limit of this infinite sequence is also 1.

Example (3).

- *Let p be a positive prime number and $b \in (\sqrt{p}, +\infty)$. The infinite sequence $\{a_n\}_{n=0}^{\infty}$ defined recursively by*

$$\begin{cases} a_0 & = & b \\ a_{n+1} & = & \dfrac{1}{2}(a_n + \dfrac{\alpha^2}{a_n}) \quad \text{for any} \quad n \in \mathbb{N} \end{cases}$$

  is decreasing and bounded below in $\mathbb{R}$.
  The infimum of its set of all terms is $\sqrt{p}$. Coincidentally, the limit of this infinite sequence is also $\sqrt{p}$.

The 'coincidence' in these examples is no isolated phenomenon. It is a consequence of the **Bounded-Monotone Theorem for infinite sequences of real numbers**.

7. **Bounded-Monotone Theorem for infinite sequences of real numbers.**

   Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers. Denote the set of all terms of $\{a_n\}_{n=0}^{\infty}$ by T.

   Suppose $\{a_n\}_{n=0}^{\infty}$ is $\left\{ \begin{array}{c} \text{increasing} \\ \text{decreasing} \end{array} \right\}$.

   Further suppose $\{a_n\}_{n=0}^{\infty}$ is $\left\{ \begin{array}{c} \text{bounded above} \\ \text{bounded below} \end{array} \right\}$ in $\mathbb{R}$. Denote the $\left\{ \begin{array}{c} \text{supremum} \\ \text{infimum} \end{array} \right\}$ of T in $\mathbb{R}$ by σ, if it exists.

   Then $\left\{ \begin{array}{c} \sup(T) \\ \inf(T) \end{array} \right\}$ exists in $\mathbb{R}$, $\{a_n\}_{n=0}^{\infty}$ converges in $\mathbb{R}$, and $\lim\limits_{n\to\infty} a_n = \sigma$.

   (Furthermore, for any $\left\{ \begin{array}{c} \text{upper bound} \\ \text{lower bound} \end{array} \right\}$ β of the infinite sequence $\{a_n\}_{n=0}^{\infty}$, the inequality $\left\{ \begin{array}{c} \sigma \leq \beta \\ \sigma \geq \beta \end{array} \right\}$ holds. Also, for any $k \in \mathbb{N}$, the inequality $\left\{ \begin{array}{c} a_k \leq \sigma \\ a_k \geq \sigma \end{array} \right\}$ holds.)

   Remark. The Bounded-Monotone Theorem is a consequence of the **Least-upper-bound Axiom**:

   Let A be a non-empty subset of $\mathbb{R}$. Suppose A is bounded above in $\mathbb{R}$. Then A has a least upper bound in $\mathbb{R}$.

8. **Appendix: Definition for limit of sequence, and a proof for the Bounded-Monotone Theorem.**

   To give a satisfactory argument for the Bounded-Monotone Theorem, we first need to formulate a satisfactory definition for the notion of limit of sequence.

   Definition.

   Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers, and ℓ be a real number.

   We say that $\{a_n\}_{n=0}^{\infty}$ converges to ℓ, and write $\lim\limits_{n\to\infty} a_n = \ell$ if the condition $(\star)$ is satisfied:

   $(\star)$ *For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, if $k > N$ then $|a_k - \ell| < \varepsilon$.*

   Proof of the Bounded-Monotone Theorem.

   Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers. Suppose $\{a_n\}_{n=0}^{\infty}$ is increasing, and is bounded above in $\mathbb{R}$.

   Define $T = \{x \in \mathbb{R} : x = a_n \text{ for some } n \in \mathbb{N}\}$.

   Note that $a_0 \in T$. Then $T \neq \emptyset$.

   By assumption T is bounded above in $\mathbb{R}$. Then by the Least-upper-bound Axiom, T has a supremum in $\mathbb{R}$. Write $\sigma = \sup(T)$.

   We verify that $\{a_n\}_{n=0}^{\infty}$ converges to σ:

   - Pick any $\varepsilon > 0$.

     [Can we name an appropriate natural number N for which it happens that whenever $k > N$, $|a_k - \sigma| < \varepsilon$? How?]

     Note that $\sigma - \varepsilon < \sigma$.
     Then by definition, $\sigma - \varepsilon$ is not an upper bound of T in $\mathbb{R}$.
     Therefore there exists some $x \in T$ such that $x > \sigma - \varepsilon$.
     For the same x, there exists some $N \in \mathbb{N}$ such that $x = a_N$.

     [Ask: Is it true that whenever $k > N$, $|a_k - \sigma| < \varepsilon$?]

     Pick any $k \in \mathbb{N}$. Suppose $k > N$.
     Then we have $a_k > a_N = x > \sigma - \varepsilon$ by assumption.
     Therefore $|a_k - \sigma| = \sigma - a_k < \varepsilon$.

   The result follows.