1. **Definition.**

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{R}$.

(a) Let $\kappa \in \mathbb{R}$.

κ *is said to be* $\left\{\begin{array}{c}\textbf{upper bound}\\ \textbf{lower bound}\end{array}\right\}$ **of** $\{a_n\}_{n=0}^{\infty}$ **in** $\mathbb{R}$ *if, for any* $n \in \mathbb{N}$, $\left\{\begin{array}{c} a_n \leq \kappa \\ a_n \geq \kappa \end{array}\right\}$.

(b) $\{a_n\}_{n=0}^{\infty}$ *is said to be* $\left\{\begin{array}{c}\textbf{bounded above}\\ \textbf{bounded below}\end{array}\right\}$ **in** $\mathbb{R}$ *if there exists some* $\kappa \in \mathbb{R}$ *such that for any* $n \in \mathbb{N}$, $\left\{\begin{array}{c} a_n \leq \kappa \\ a_n \geq \kappa \end{array}\right\}$.

Bounded-ness for infinite sequences of real numbers can be re-formulated in terms of bounded-ness for their corresponding 'sets of all terms'.

Lemma (1).

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{R}$. Define

$$T(\{a_n\}_{n=0}^{\infty}) = \{x \in \mathbb{R} : x = a_n \text{ for some } n \in \mathbb{N}\}.$$

(It is the set of all terms of $\{a_n\}_{n=0}^{\infty}$.)

The statements below hold:

(a) $\{a_n\}_{n=0}^{\infty}$ *is bounded above in* $\mathbb{R}$ *by* β *iff* $T(\{a_n\}_{n=0}^{\infty})$ *is bounded above in* $\mathbb{R}$ *by* β.

(b) $\{a_n\}_{n=0}^{\infty}$ *is bounded below in* $\mathbb{R}$ *by* β *iff* $T(\{a_n\}_{n=0}^{\infty})$ *is bounded below in* $\mathbb{R}$ *by* β.

2. **Definition.**

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{R}$.

(a) *$\{a_n\}_{n=0}^{\infty}$ is said to be* $\left\{\begin{array}{c}\textbf{increasing}\\ \textbf{decreasing}\end{array}\right\}$ *if, for any $n \in \mathbb{N}$,* $\left\{\begin{array}{c} a_n \leq a_{n+1} \\ a_n \geq a_{n+1}\end{array}\right\}$.

(b) *$\{a_n\}_{n=0}^{\infty}$ is said to be* $\left\{\begin{array}{c}\textbf{strictly increasing}\\ \textbf{strictly decreasing}\end{array}\right\}$ *if, for any $n \in \mathbb{N}$,* $\left\{\begin{array}{c} a_n < a_{n+1} \\ a_n > a_{n+1}\end{array}\right\}$.

Remarks on terminology.

(a) $\{a_n\}_{n=0}^{\infty}$ is said to be **monotonic** if $\{a_n\}_{n=0}^{\infty}$ is increasing or decreasing.

(b) $\{a_n\}_{n=0}^{\infty}$ is said to be **strictly monotonic** if $\{a_n\}_{n=0}^{\infty}$ is strictly increasing or strictly decreasing.

Lemma (2).

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{R}$.

Define $T(\{a_n\}_{n=0}^{\infty}) = \{x \in \mathbb{R} : x = a_n$ for some $n \in \mathbb{N}\}$.

The statements below hold:

(a) *Suppose $\{a_n\}_{n=0}^{\infty}$ is strictly increasing. Then $T(\{a_n\}_{n=0}^{\infty})$ has no greatest element.*

(b) *Suppose $\{a_n\}_{n=0}^{\infty}$ is strictly decreasing. Then $T(\{a_n\}_{n=0}^{\infty})$ has no least element.*

3. **Example (1).**

For any $n \in \mathbb{N}$, define $a_n = \dfrac{(n+1)(n+4)}{(n+2)(n+3)}$. Define $T = \{x \mid x = a_n \text{ for some } n \in \mathbb{N}\}$.

(a) $\{a_n\}_{n=0}^{\infty}$ is bounded above in $\mathbb{R}$ by 1. (Equivalently, T is bounded above in $\mathbb{R}$ by 1.)

Proof. Let $n \in \mathbb{N}$.

$$a_n = \frac{(n+1)(n+4)}{(n+2)(n+3)} = \frac{n^2+5n+4}{n^2+5n+6} = 1 - \frac{2}{n^2+5n+6} \leq 1 - 0 = 1.$$

Hence $\{a_n\}_{n=0}^{\infty}$ is bounded above in $\mathbb{R}$ by 1. □

(b) $\{a_n\}_{n=0}^{\infty}$ is strictly increasing.

Proof. Let $n \in \mathbb{N}$. [Hope to deduce: $a_{n+1} > a_n$.]

$$a_{n+1} - a_n = \cdots = \frac{4}{(n+2)(n+3)(n+4)} > 0$$

Then $a_{n+1} > a_n$. □

(c) T has no greatest element.

Proof. [Proof-by-contradiction argument.]

[Ask: Is there any element of T which is greater than λ?] →

Suppose T had a greatest element, say, λ.

By definition, $\lambda \in T$.

By definition of T, there exists some $n \in \mathbb{N}$ such that $\lambda = a_n$.

Define $x_0 = a_{n+1}$.

By definition of T, $x_0 \in T$.

Also, $x_0 = a_{n+1} > a_n = \lambda$.

Contradiction arises. □

(d) *For any $\beta \in \mathbb{R}$, if $\beta \geq 1$ then β is an upper bound of T in $\mathbb{R}$.* (Exercise.)

(e) *For any $\beta \in \mathbb{R}$, if $\beta < 1$ then β is not an upper bound of T in $\mathbb{R}$.*

Remark. We assume the validity of the statement (AP) below (which is known as the Archimedean Principle):

(AP) *For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.*

Proof.

Pick any $\beta \in \mathbb{R}$. Suppose $\beta < 1$.

[Apply the proof-by-contradiction method to prove that β is not an upper bound of T in $\mathbb{R}$.]

- Suppose β were an upper bound of T in $\mathbb{R}$.

[Ask: Is there any element of T greater than β?]

Define $\varepsilon = \frac{1-\beta}{2}$. By definition, $\varepsilon > 0$.

Then, by (AP), there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.

For the same ε, N, we have $(N^2+5N+6)\varepsilon \geq N\varepsilon > 1$.

Then $\frac{2}{N^2+5N+6} < 2\varepsilon = 1-\beta$. Therefore $\beta < 1 - \frac{2}{N^2+5N+6} = a_N$.

But β was an upper bound of T in $\mathbb{R}$. Contradiction arises.

Roughwork.

Re-formulate this question:

Can we name an appropriate $k \in \mathbb{N}$ for which '$a_k > \beta$' holds?

So we study the inequality '$a_m > \beta$' with 'indeterminate' m.

Ask: Can we re-formulate this inequality as '$P(m)\,Q(\beta) > 1$' in which $P(m)$ is some integer depending on m but not β, and $Q(\beta)$ is some positive number depending on β but not m? (This will suggest a hint on how to use (AP).)

Answer (after some work). $(m^2+5m+6)\cdot\frac{1-\beta}{2} > 1$.

(f) *T has a supremum in $\mathbb{R}$, namely 1.*

Proof.

The set of all upper bounds of T in $\mathbb{R}$ is $[1, +\infty)$, whose least element is 1.

4. **Example (2).** [This example provides the reason why $0.\dot{9} = 1$, with $0.\dot{9}$ being understood as the limit of the infinite sequence $\left\{ \sum_{k=0}^{n} \frac{9}{10^{k+1}} \right\}_{n=0}^{\infty}$.]

For any $n \in \mathbb{N}$, define $a_n = \sum_{k=0}^{n} \frac{9}{10^{k+1}}$. *Define* $T = \{x \mid x = a_n \text{ for some } n \in \mathbb{N}\}$.

(a) *$\{a_n\}_{n=0}^{\infty}$ is bounded above in $\mathbb{R}$ by 1. (Equivalently, T is bounded above in $\mathbb{R}$ by 1.)*

Proof.

Let $n \in \mathbb{N}$. We have

$$a_n = \sum_{k=0}^{n} \frac{9}{10^{k+1}} = \frac{9}{10} \cdot \frac{1 - 1/10^{n+1}}{1 - 1/10} = 1 - \frac{1}{10^{n+1}} \leq 1 - 0 = 1.$$

Hence $\{a_n\}_{n=0}^{\infty}$ is bounded above by 1.

(b) *$\{a_n\}_{n=0}^{\infty}$ is strictly increasing.*

Proof.

Let $n \in \mathbb{N}$. We have

$$a_{n+1} - a_n = \sum_{k=0}^{n+1} \frac{9}{10^{k+1}} - \sum_{k=0}^{n} \frac{9}{10^{k+1}} = \frac{9}{10^{n+2}} > 0.$$

Then $a_{n+1} > a_n$.

(c) *T has no greatest element.* (Exercise.)

(d) *For any* $\beta \in \mathbb{R}$, *if* $\beta \geq 1$ *then* β *is an upper bound of* T *in* $\mathbb{R}$. (Exercise.)

(e) *For any* $\beta \in \mathbb{R}$, *if* $\beta < 1$ *then* β *is not an upper bound of* T *in* $\mathbb{R}$.

Remark. We assume the validity of the statement (AP) below (which is known as the Archimedean Principle):

(AP) *For any* $\varepsilon > 0$, *there exists some* $N \in \mathbb{N}\backslash\{0\}$ *such that* $N\varepsilon > 1$.

Proof.

Pick any $\beta \in \mathbb{R}$. Suppose $\beta < 1$.

[Apply the proof-by-contradiction method to prove that β is not an upper bound of T in $\mathbb{R}$.]

- Suppose β were an upper bound of T in $\mathbb{R}$.

[Ask: Is there any element of T greater than β?]

Define $\varepsilon = 1 - \beta$. By definition $\varepsilon > 0$.

Then, by (AP), there exists some $N \in \mathbb{N}\backslash\{0\}$ such that $N\varepsilon > 1$.

For the same ε, N, we have $10^{N+1}\varepsilon \geq N\varepsilon > 1$.

Then $\frac{1}{10^{N+1}} < \varepsilon = 1 - \beta$. Therefore $\beta < 1 - \frac{1}{10^{N+1}} = \sum_{k=0}^{N} \frac{9}{10^{k+1}} = a_N$.

But β was an upper bound of T in $\mathbb{R}$. Contradiction arises.

Roughwork.

Re-formulate this question:

Can we name an appropriate $k \in \mathbb{N}$ for which '$a_k > \beta$' holds?

So we study the inequality '$a_m > \beta$' with indeterminate m.

Ask: Can we re-formulate this inequality as '$P(m)\,Q(\beta) > 1$' in which $P(m)$ is some integer depending on m but not β, and $Q(\beta)$ is some positive number depending on β but not m? (This will suggest a hint on how to use (AP).)

Answer (after some work).

$10^{m+1} \cdot (1-\beta) > 1$.

(f) *T has a supremum in* $\mathbb{R}$, *namely* 1. (Exercise.)

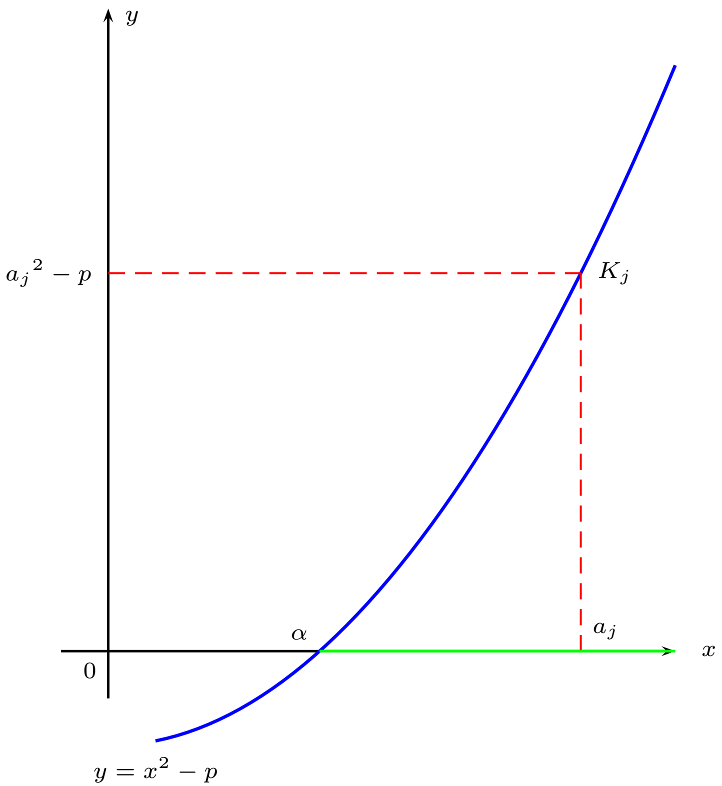
This is the reason why $0.\dot{9} = 1$.

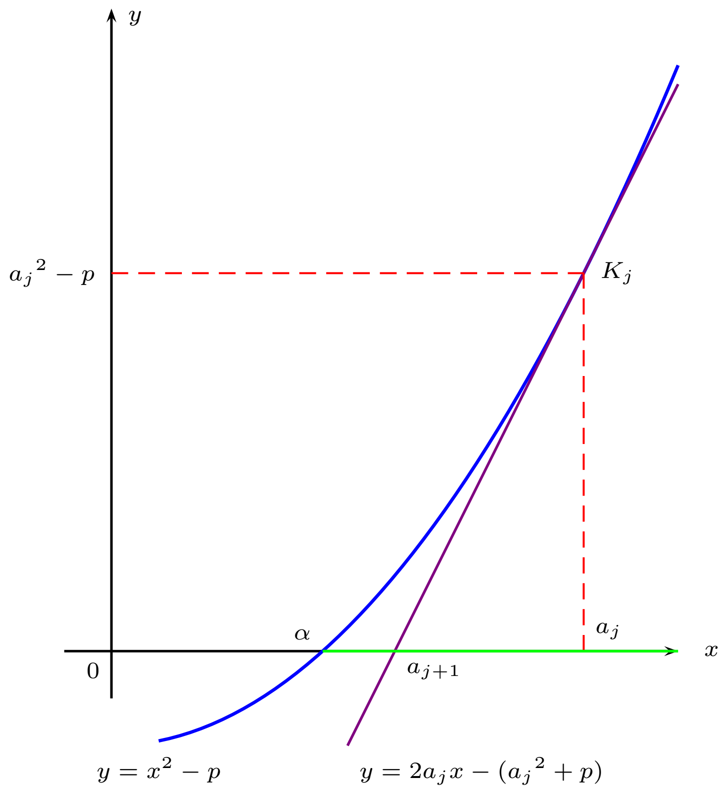
5. **Example (3).**

Let p be a positive prime number. Define $\alpha = \sqrt{p}$. Let $b \in (\alpha, +\infty)$. Let $\{a_n\}_{n=0}^{\infty}$ be the infinite sequence defined recursively by

$$\begin{cases} a_0 = b \\ a_{n+1} = \frac{1}{2}(a_n + \frac{\alpha^2}{a_n}) & \textit{for any} \quad n \in \mathbb{N} \end{cases}$$

$\{a_n\}_{n=0}^{\infty}$ provides 'better and better' approximations for $\alpha = \sqrt{p}$:





Example (3).

Let p be a positive prime number. Define $\alpha = \sqrt{p}$. Let $b \in (\alpha, +\infty)$.

Let $\{a_n\}_{n=0}^{\infty}$ be the infinite sequence defined recursively by

$$\begin{cases} a_0 & = b \\ a_{n+1} & = \frac{1}{2}(a_n + \frac{\alpha^2}{a_n}) \quad \text{for any} \quad n \in \mathbb{N} \end{cases}$$

Define $T = \{x \in \mathbb{R} : x = a_n \text{ for some } n \in \mathbb{N}\}$.

(a) *For any $n \in \mathbb{N}$, $a_n > \alpha$.*

Remark. As a consequence, $\{a_n\}_{n=0}^{\infty}$ is bounded below in $\mathbb{R}$ by α, and T is bounded below in $\mathbb{R}$ by α.

(b) *$\{a_n\}_{n=0}^{\infty}$ is strictly decreasing.*

(c) $0 < a_n - \alpha < \dfrac{b - \alpha}{2^n}$ *for any $n \in \mathbb{N} \backslash \{0\}$.*

Remark. Heuristically speaking, the infinite sequence $\{a_n\}_{n=0}^{\infty}$ will descend to as close as α as we like, but it will never 'reach' α.

(d) *T has no least element.*

(e) *For any $\beta \in \mathbb{R}$, if $\beta < \alpha$ then β is a lower bound of T in $\mathbb{R}$.*

(f) *For any $\beta \in \mathbb{R}$, if $\beta > \alpha$ then β is not a lower bound of T in $\mathbb{R}$.*

(g) *T has an infimum in $\mathbb{R}$, namely, α.*

Further remarks.

(1) Whether '*p is a prime number*' or not is immaterial.

(2) How about finding cubic roots of positive real numbers?
Suppose p is a positive real number and $\alpha = \sqrt[3]{p}$. Suppose $b \in (\alpha, +\infty)$. Define infinite sequence $\{a_n\}_{n=0}^{\infty}$ recursively by

$$\begin{cases} a_0 &= b \\ a_{n+1} &= \dfrac{1}{3}(2a_n + \dfrac{\alpha^3}{{a_n}^2}) \quad \textit{for any} \quad n \in \mathbb{N} \end{cases}$$

$\{a_n\}_{n=0}^{\infty}$ will provide 'better and better' approximations for α.

(3) How about finding quartic roots of positive real numbers? Quintic roots? n-th roots?

(4) The idea and method described here is a 'concrete' example of the application of **Newton's Method (for finding approximate solutions of equations)**.

6. **Some 'coincidence' in Examples (1), (2), (3).**

We make some observations on Examples (1), (2), (3).

Example (1).

- *The infinite sequence* $\left\{\dfrac{(n+1)(n+4)}{(n+2)(n+3)}\right\}_{n=0}^{\infty}$ *is increasing and bounded above in* $\mathbb{R}$.
  The supremum of its set of all terms is 1.
  Coincidentally, the limit of this infinite sequence is also 1.

Example (2).

- *The infinite sequence* $\left\{\displaystyle\sum_{k=0}^{n}\frac{9}{10^{k+1}}\right\}_{n=0}^{\infty}$ *is increasing and bounded above in* $\mathbb{R}$.
  The supremum of its set of all terms is 1.
  Coincidentally, the limit of this infinite sequence is also 1.

Some 'coincidence' in Examples (1), (2), (3).

Example (1). ...

Example (2). ...

Example (3).

- *Let p be a positive prime number and $b \in (\sqrt{p}, +\infty)$. The infinite sequence $\{a_n\}_{n=0}^{\infty}$ defined recursively by*

$$\begin{cases} a_0 &= b \\ a_{n+1} &= \frac{1}{2}(a_n + \frac{\alpha^2}{a_n}) \quad \text{for any} \quad n \in \mathbb{N} \end{cases}$$

is decreasing and bounded below in $\mathbb{R}$.
The infimum of its set of all terms is $\sqrt{p}$.
Coincidentally, the limit of this infinite sequence is also $\sqrt{p}$.

The 'coincidence' in these examples is no isolated phenomenon. It is a consequence of the **Bounded-Monotone Theorem for infinite sequences of real numbers**.

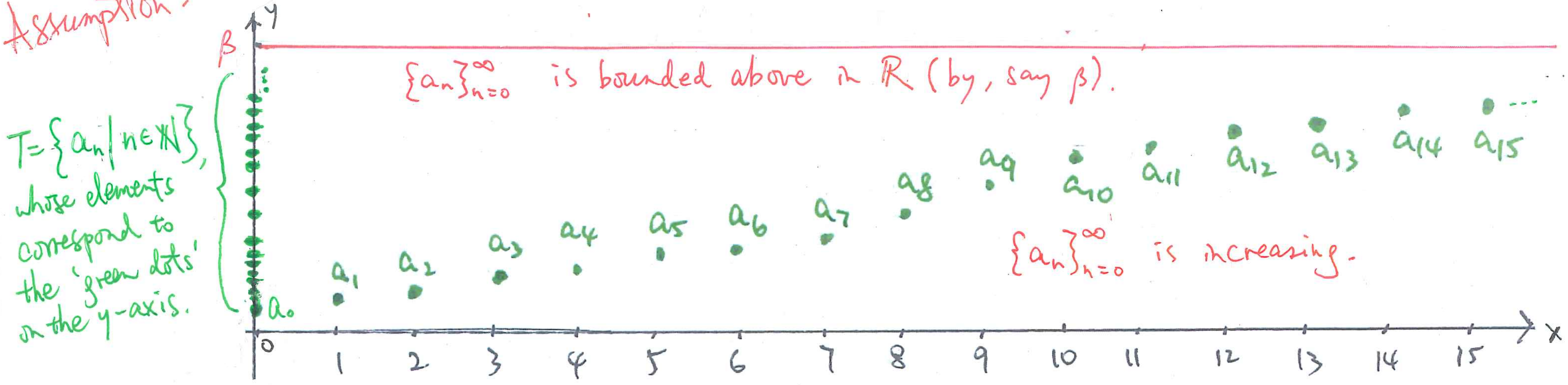
7. **Bounded-Monotone Theorem for infinite sequences of real numbers.**

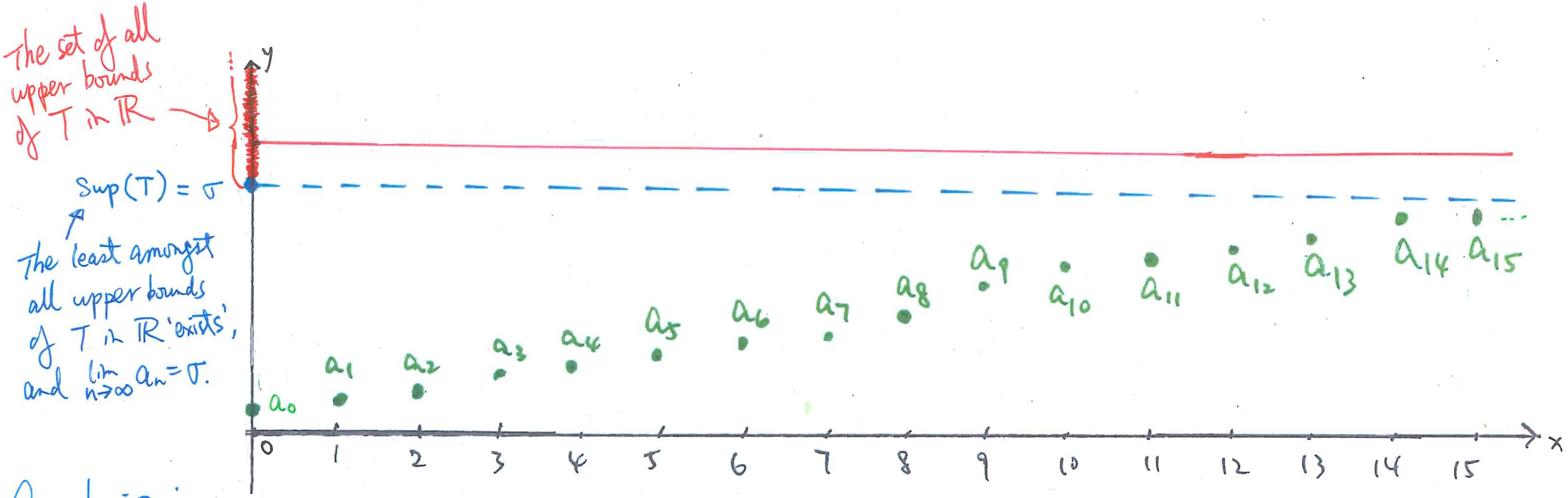
Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers.
Denote the set of all terms of $\{a_n\}_{n=0}^{\infty}$ by T.
Suppose $\{a_n\}_{n=0}^{\infty}$ is $\left\{\begin{array}{l}\textit{increasing}\\\textit{decreasing}\end{array}\right\}$. Further suppose $\{a_n\}_{n=0}^{\infty}$ is $\left\{\begin{array}{l}\textit{bounded above}\\\textit{bounded below}\end{array}\right\}$ in $\mathbb{R}$.
Denote the $\left\{\begin{array}{l}\textit{supremum}\\\textit{infimum}\end{array}\right\}$ of T in $\mathbb{R}$ by σ, if it exists.
Then $\left\{\begin{array}{c}\sup(T)\\\inf(T)\end{array}\right\}$ exists in $\mathbb{R}$, $\{a_n\}_{n=0}^{\infty}$ converges in $\mathbb{R}$, and $\lim\limits_{n\to\infty} a_n = \sigma$.
(Furthermore, for any $\left\{\begin{array}{l}\textit{upper bound}\\\textit{lower bound}\end{array}\right\}$ β of the infinite sequence $\{a_n\}_{n=0}^{\infty}$, the inequality $\left\{\begin{array}{c}\sigma \leq \beta\\\sigma \geq \beta\end{array}\right\}$ holds. Also, for any $k \in \mathbb{N}$, the inequality $\left\{\begin{array}{c}a_k \leq \sigma\\a_k \geq \sigma\end{array}\right\}$ holds.)

Remark. The Bounded-Monotone Theorem is a consequence of the **Least-upper-bound Axiom**:
Let A be a non-empty subset of $\mathbb{R}$. Suppose A is bounded above in $\mathbb{R}$. Then A has a least upper bound in $\mathbb{R}$.

Bounded - Monotone Theorem (for increasing sequences which are bounded above)

Assumption:





Conclusion:

8. Appendix: Definition for limit of sequence, and a proof for the Bounded-Monotone Theorem.

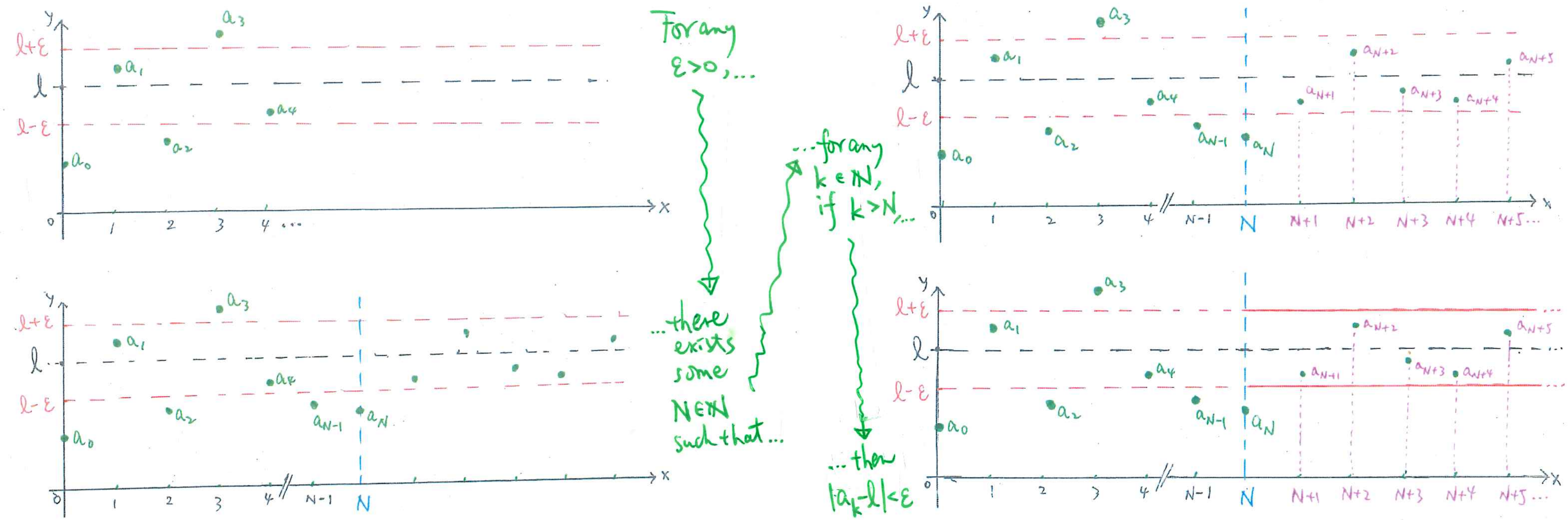
To give a satisfactory argument for the Bounded-Monotone Theorem, we first need to formulate a satisfactory definition for the notion of limit of sequence.

Definition.

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers, and ℓ be a real number.

We say that $\{a_n\}_{n=0}^{\infty}$ converges to ℓ, and write $\lim_{n\to\infty} a_n = \ell$ if the condition $(\star)$ is satisfied:

$(\star)$ *For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, if $k > N$ then $|a_k - \ell| < \varepsilon$.*



Proof of the Bounded-Monotone Theorem.

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers. Suppose $\{a_n\}_{n=0}^{\infty}$ is increasing, and is bounded above in $\mathbb{R}$.

Define $T = \{x \in \mathbb{R} : x = a_n \text{ for some } n \in \mathbb{N}.\}$.

Note that $a_0 \in T$. Then $T \neq \emptyset$.

By assumption, T is bounded above in $\mathbb{R}$. (Why? Apply Lemma (1).)

Then, by the Least-upper-bound Axiom, T has a supremum in $\mathbb{R}$.

Write $\sigma = \sup(T)$.

We verify that $\{a_n\}_{n=0}^{\infty}$ converges to σ:

[What we want to verify is: 'For any $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, if $k > N$ then $|a_k - \sigma| < \varepsilon$.']

- Pick any $\varepsilon > 0$. [Ask: Can we name an appropriate $N \in \mathbb{N}$ for which whenever $k > N$, $|a_k - \sigma| < \varepsilon$? How?]

Note that $\sigma - \varepsilon < \sigma$. Then, by definition, $\sigma - \varepsilon$ is not an upper bound of T in $\mathbb{R}$.

Therefore, there exists some $x \in T$ such that $x > \sigma - \varepsilon$.

For the same x, there exists some $N \in \mathbb{N}$ such that $x = a_N$.

[Ask: Is it true that whenever $k > N$, $|a_k - \sigma| < \varepsilon$?]

Pick any $k \in \mathbb{N}$. Suppose $k > N$.

Then $a_k > a_N = x > \sigma - \varepsilon$ by assumption.

Therefore $|a_k - \sigma| = \sigma - a_k < \varepsilon$. □