1. **Definition.**

Let S be a subset of $\mathbb{R}$.

(a) *Let $\lambda \in S$.*

λ *is said to be a* $\left\{ \begin{array}{c} \textbf{greatest} \\ \textbf{least} \end{array} \right\}$ **element** *of S if, for any $x \in S$,* $\left\{ \begin{array}{c} x \leq \lambda \\ x \geq \lambda \end{array} \right\}$.

(b) *S is said to have a* $\left\{ \begin{array}{c} \textit{greatest} \\ \textit{least} \end{array} \right\}$ *element if there exists some $\lambda \in S$ such that for any $x \in S$,* $\left\{ \begin{array}{c} x \leq \lambda \\ x \geq \lambda \end{array} \right\}$.

Remarks.

(A) 'If it exists then it is unique': Suppose λ, λ' are $\left\{ \begin{array}{c} \text{greatest} \\ \text{least} \end{array} \right\}$ elements of S. Then $\lambda = \lambda'$.

(B) **Notation.** We denote the $\left\{ \begin{array}{c} \text{greatest} \\ \text{least} \end{array} \right\}$ element of S by $\left\{ \begin{array}{c} \max(S) \\ \min(S) \end{array} \right\}$.

(C) **Terminology.** We may choose to write

'*S has a greatest element*'

as

'$\max(S)$ *exists*'.

Et cetera. The situation is analogous for least element.

2. **Definition.**

Let S be a subset of $\mathbb{R}$.

(a) *Let $\beta \in \mathbb{R}$.*

β *is said to be a(n)* $\left\{ \begin{array}{c} \textbf{upper} \\ \textbf{lower} \end{array} \right\}$ **bound of** S **in** $\mathbb{R}$ *if, for any* $x \in S$, $\left\{ \begin{array}{c} x \leq \beta \\ x \geq \beta \end{array} \right\}$.

(b) S *is said to be* **bounded** $\left\{ \begin{array}{c} \textbf{above} \\ \textbf{below} \end{array} \right\}$ **in** $\mathbb{R}$ *if there exists some* $\beta \in \mathbb{R}$ *such that*

for any $x \in S$, $\left\{ \begin{array}{c} x \leq \beta \\ x \geq \beta \end{array} \right\}$.

(c) S is said to be **bounded in** $\mathbb{R}$ if S is bounded above in $\mathbb{R}$ and bounded below in $\mathbb{R}$.

Remarks.

(A) If S has one upper bound then it has infinitely many upper bounds.
It does not make sense to write '*the* upper bound of S is so-and-so'.
The situation is similar for 'being bounded below'.

(B) *Suppose λ is a greatest element of S. Then λ is an upper bound of S.*
(How about its converse?)
The situation is similar for 'least element' and 'lower bound'.

3. **Example (A). (Well-ordering Principle for Integers.)**

Recall this statement below, known as the Well-ordering Principle for Integers (WOPI):

Let S be a non-empty subset of $\mathbb{N}$. S has a least element.

There are various re-formulations of the statement (WOPI):

- (WOPIL) *Let T be a non-empty subset of $\mathbb{Z}$.*
  Suppose T is bounded below in $\mathbb{R}$ by some $\beta \in \mathbb{Z}$.
  Then T has a least element.
- (WOPIG) *Let U be a non-empty subset of $\mathbb{Z}$.*
  Suppose U is bounded above in $\mathbb{R}$ by some $\gamma \in \mathbb{Z}$.
  Then U has a greatest element.

(The proof for the logical equivalence of (WOPI), (WOPIL), (WOPIG) is left as an exercise.)

From now on all three of them are referred to as the Well-ordering Principle for Integers.

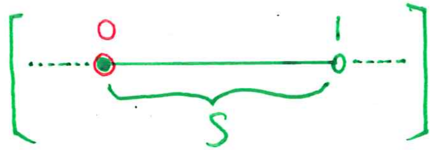
4. **Example (B).**

S	least element?	greatest element?	bounded below in $\mathbb{R}$?	bounded above in $\mathbb{R}$?
$[0,1)$	0 *Aa*	nil *Ab*	*Ac*	Yes (by 1) *Ad*
$[0,+\infty)$	*Ba*	nil *Bb*	*Bc*	No *Bd*
$(0,+\infty)$	*Ca*	*Cb*	*Cc*	*Cd*
$[0,1)\cap\mathbb{Q}$	*Da*	*Db*	*Dc*	*Dd*
$[0,+\infty)\cap\mathbb{Q}$	*Ea*	*Eb*	*Ec*	*Ed*
$[0,1)\setminus\mathbb{Q}$	*Fa*	*Fb*	*Fc*	*Fd*
$[0,+\infty)\setminus\mathbb{Q}$	*Ga*	*Gb*	*Gc*	*Gd*

(Aa): $[0,1)$ *has a least element, namely,* 0.

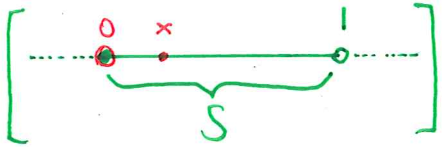
Proof. Write $S = [0,1)$.

- Note that $0 \in S$.



[Check: For any $x \in S$, $x \geq 0$.]

- Pick any $x \in S$.
  By definition of S, $0 \leq x < 1$.
  In particular, $x \geq 0$.

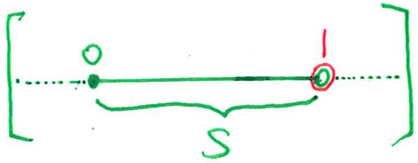


It follows that 0 is a least element of S. □

(Ad): $[0,1)$ *is bounded above in* $\mathbb{R}$ *by* 1.

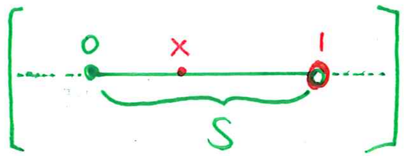
Proof. Write $S = [0,1)$.

- Note that $1 \in \mathbb{R}$.



[Check: For any $x \in S$, $x \leq 1$.]

- Pick any $x \in S$.
  By definition of S, $0 \leq x < 1$.
  In particular, $x < 1$.
  Then $x \leq 1$.



It follows that S is bounded above in $\mathbb{R}$ by 1. □

(Ab): $[0, 1)$ *has no greatest element.*

Proof. [Proof-by-contradiction argument.]

Write $S = [0, 1)$.

Suppose S had a greatest element, say, λ.

Define $x_0 = \dfrac{\lambda + 1}{2}$.

Then $0 \leq \lambda < x_0 < 1$.

We have $x_0 \in S$ and $x_0 > \lambda$.

Contradiction arises. $\square$

(Bb): $[0, +\infty)$ *has no greatest element.*

Proof. [Proof-by-contradiction argument.]

Write $S = [0, +\infty)$.

Suppose S had a greatest element, say λ.

Since $\lambda \in S$, we have $\lambda \geq 0$.

Define $x_0 = \lambda + 1$.

Then $0 \leq \lambda < x_0$.

We have $x_0 \in S$ and $x_0 > \lambda$.

Contradiction arises. $\square$

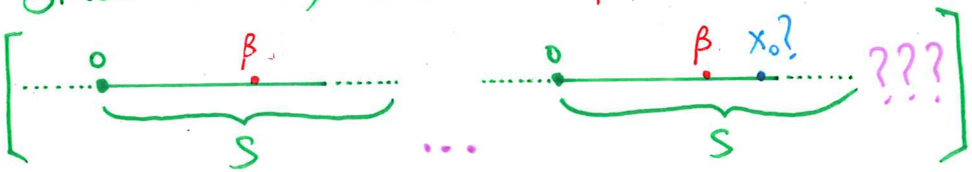
(Bd): $[0, +\infty)$ *is not bounded above in* $\mathbb{R}$.

Proof. [Proof-by-contradiction argument.]

Write $S = [0, +\infty)$.

Suppose S were bounded above in $\mathbb{R}$, say, by β.

Since $0 \in S$, we have $\beta \geq 0$.



Define $x_0 = \beta + 1$.

Then $0 \leq \beta < x_0$.

We have $x_0 \in S$ and $x_0 > \beta$.

Contradiction arises. $\square$

Example (B).

S	least element?	greatest element?	bounded below in $\mathbb{R}$?	bounded above in $\mathbb{R}$?
$[0,1)$	0 Aa	nil Ab	↑ Ac	Yes (by 1) Ad
$[0,+\infty)$	0 Ba	nil Bb	All Bc	No Bd
$(0,+\infty)$	nil Ca	nil Cb	yes Cc	No Cd
$[0,1)\cap\mathbb{Q}$	0 Da	nil Db	(by Dc	Yes (by 1) Dd
$[0,+\infty)\cap\mathbb{Q}$	0 Ea	nil Eb	0) Ec	No Ed
$[0,1)\setminus\mathbb{Q}$	nil Fa	nil Fb	Fc	Yes (by 1) Fd
$[0,+\infty)\setminus\mathbb{Q}$	nil Ga	nil Gb	↓ Gc	No Gd

5. **Example (C).**

Let

$$S = \{x \in \mathbb{R} : x^2 \leq (\sqrt{2}+1)x - \sqrt{2}\},$$

and $T = S \backslash \mathbb{Q}$.

(S is in fact the solution set of the inequality

$$x^2 \leq (\sqrt{2}+1)x - \sqrt{2}$$

with unknown x in the reals.)

(a) *S has a greatest element and S has a least element.*

Proof.

- Note that $S = [1, \sqrt{2}]$.
  S has a greatest element, namely $\sqrt{2}$.
  S has a least element, namely 1.

(b) *S is bounded above and below in* $\mathbb{R}$.

Proof.

- S has a least element and a greatest element.
  They are respectively a lower bound and an upper bound of S in $\mathbb{R}$.

Example (C).

Let $S = \{x \in \mathbb{R} : x^2 \leq (\sqrt{2}+1)x - \sqrt{2}\}$, *and* $T = S \backslash \mathbb{Q}$.

(c) T *has a greatest element, and* T *has no least element.*

Proof.

- Note that $T = [1, \sqrt{2}] \backslash \mathbb{Q}$.
- We have $\sqrt{2} \in [1, \sqrt{2}]$, and $\sqrt{2}$ is irrational. Then $\sqrt{2} \in T$.
  Pick any $x \in T$.
  By definition, $1 \leq x \leq \sqrt{2}$ and x is irrational. In particular $x \leq \sqrt{2}$.
  Therefore, by definition, $\sqrt{2}$ is a greatest element of T.
- Suppose T had a least element, say, λ. By definition, λ is irrational and $1 \leq \lambda \leq \sqrt{2}$.
  Since λ is irrational, $\lambda \neq 1$. Then $\lambda > 1$.
  Define $x_0 = \dfrac{1+\lambda}{2}$. By definition, $1 < x_0 < \lambda \leq \sqrt{2}$.
  Moreover x_0 is irrational. (Why? Fill in the detail.)
  Then $x_0 \in T$. But $x_0 < \lambda$ and λ is a least element of T. Contradiction arises.

(d) T *is bounded above and below in* $\mathbb{R}$.

Proof.

- T has a least element. It is a lower bound of T in $\mathbb{R}$.
- By definition, for any $x \in T$, $x \leq \sqrt{2}$. Then $\sqrt{2}$ is an upper bound of T in $\mathbb{R}$.

6. **Example (D).**

Let $S = \left\{ \frac{1}{m+1} + \frac{1}{n+1} \;\middle|\; m, n \in \mathbb{N} \right\}$.

(a) *S has a greatest element and S has no least element.*

(b) *S is bounded above and below in $\mathbb{R}$.*

(Heuristically, S '=' $\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \ldots;$
$1, \frac{5}{6}, \frac{3}{4}, \frac{7}{10}, \ldots;$
$\frac{2}{3}, \frac{7}{12}, \frac{8}{15}, \ldots;$
$\frac{1}{2}, \frac{9}{20}, \ldots;$
$\frac{2}{5}, \ldots;$
$\ddots \}$)

Proof of (a).

- We verify that S has a greatest element, namely 2:
  - * Note that $2 \in S$. (Reason: $2 = \frac{1}{0+1} + \frac{1}{0+1}$, and $0 \in \mathbb{N}$.)
  - * Pick any $x \in S$.

    [Check: For any $x \in S$, $x \leq 2$.]

    By the definition of S, there exists some $m, n \in \mathbb{N}$ such that $x = \frac{1}{m+1} + \frac{1}{n+1}$.

    Since $m, n \in \mathbb{N}$, we have $m \geq 0$ and $n \geq 0$.

    Then $x = \frac{1}{m+1} + \frac{1}{n+1} \leq \frac{1}{0+1} + \frac{1}{0+1} = 2$.
  - * It follows that 2 is a greatest element of S. □

Example (D).

Let $S = \left\{ \frac{1}{m+1} + \frac{1}{n+1} \;\middle|\; m, n \in \mathbb{N} \right\}$.

(a) *S has a greatest element and S has no least element.*

(b) *S is bounded above and below in* $\mathbb{R}$.

Proof of (a).

[We apply the proof-by-contradiction method to prove that S has no least element.]

Suppose S had a least element, say, λ.

By definition, $\lambda \in S$.

[Ask: Is there any element of S which is less than λ? How to conceive it?]

By the definition of S, there exist some $m_0, n_0 \in \mathbb{N}$ such that $\lambda = \frac{1}{m_0+1} + \frac{1}{n_0+1}$.

Define $x_0 = \frac{1}{m_0+2} + \frac{1}{n_0+1}$. By the definition of S, $x_0 \in S$. (Why?)

Also, $x_0 = \frac{1}{m_0+2} + \frac{1}{n_0+1} < \frac{1}{m_0+1} + \frac{1}{n_0+1} = \lambda$. (Why?)

Contradiction arises. □

7. **Example (B) re-visited for a special observation.**

Consider each subset S of $\mathbb{R}$ studied in Example (B):

(a) If S is bounded below in $\mathbb{R}$, then its lower bounds seem to form a closed interval of the form $(-\infty, \rho]$, which has a greatest element, namely ρ.
We may refer to this number ρ as the greatest amongst all lower bounds of S in $\mathbb{R}$, or simply, a greatest lower bound of S in $\mathbb{R}$.
Remark. The only 'cases' where the justification for the observation is not easy are $[0,1) \cap \mathbb{Q}$, $[0,+\infty) \cap \mathbb{Q}$.

(b) ...

S	least element?	greatest element?	bounded below in $\mathbb{R}$?	bounded above in $\mathbb{R}$?	set of all lower bounds?	set of all upper bounds?	greatest lower bound?	least upper bound?
$[0,1)$	0	*nil*	Yes (by 0)	Yes (by 1)	$(-\infty, 0]$	$[1,+\infty)$	0	1
$[0,+\infty)$	0	*nil*	Yes (by 0)	No	$(-\infty, 0]$	$\emptyset$	0	*nil*
$(0,+\infty)$	*nil*	*nil*	Yes (by 0)	No	$(-\infty, 0]$	$\emptyset$	0	*nil*
$[0,1)\cap\mathbb{Q}$	0	*nil*	Yes (by 0)	Yes (by 1)	$(-\infty, 0]$	$[1,+\infty)$	0	1
$[0,+\infty)\cap\mathbb{Q}$	0	*nil*	Yes (by 0)	No	$(-\infty, 0]$	$\emptyset$	0	*nil*
$[0,1)\backslash\mathbb{Q}$	*nil*	*nil*	Yes (by 0)	Yes (by 1)	$(-\infty, 0]$	$[1,+\infty)$	0	1
$[0,+\infty)\backslash\mathbb{Q}$	*nil*	*nil*	Yes (by 0)	No	$(-\infty, 0]$	$\emptyset$	0	*nil*

Remark. We can make similar observations on Example (C) and Example (D).

Example (B) re-visited for a special observation.

Consider each subset S of $\mathbb{R}$ studied in Example (B):

(a) ...

(b) If S is bounded above in $\mathbb{R}$, then its upper bounds seem to form a closed interval of the form $[\sigma, +\infty)$, which has a least element, namely σ.
We may refer to this number σ as the least amongst all upper bounds of S in $\mathbb{R}$, or simply, a least upper bound of S in $\mathbb{R}$.
Remark. The only 'case' where the justification for the observation is not easy is $[0, 1) \cap \mathbb{Q}$.

S	least element?	greatest element?	bounded below in $\mathbb{R}$?	bounded above in $\mathbb{R}$?	set of all lower bounds?	set of all upper bounds?	greatest lower bound?	least upper bound?
$[0, 1)$	0	*nil*	Yes (by 0)	Yes (by 1)	$(-\infty, 0]$	$[1, +\infty)$	0	1
$[0, +\infty)$	0	*nil*	Yes (by 0)	No	$(-\infty, 0]$	$\emptyset$	0	*nil*
$(0, +\infty)$	*nil*	*nil*	Yes (by 0)	No	$(-\infty, 0]$	$\emptyset$	0	*nil*
$[0, 1) \cap \mathbb{Q}$	0	*nil*	Yes (by 0)	Yes (by 1)	$(-\infty, 0]$	$[1, +\infty)$	0	1
$[0, +\infty) \cap \mathbb{Q}$	0	*nil*	Yes (by 0)	No	$(-\infty, 0]$	$\emptyset$	0	*nil*
$[0, 1) \setminus \mathbb{Q}$	*nil*	*nil*	Yes (by 0)	Yes (by 1)	$(-\infty, 0]$	$[1, +\infty)$	0	1
$[0, +\infty) \setminus \mathbb{Q}$	*nil*	*nil*	Yes (by 0)	No	$(-\infty, 0]$	$\emptyset$	0	*nil*

Remark. We can make similar observations on Example (C) and Example (D).

8. **Definition.**

Let S be a subset of $\mathbb{R}$, and σ be a real number.

Suppose S is $\left\{\begin{array}{c}\textit{bounded above}\\ \textit{bounded below}\end{array}\right\}$ *in $\mathbb{R}$, and σ is a(n)* $\left\{\begin{array}{c}\textit{upper bound}\\ \textit{lower bound}\end{array}\right\}$ *of S in $\mathbb{R}$.*

σ is said to be a(n) $\left\{\begin{array}{c}\textbf{supremum}\\ \textbf{infimum}\end{array}\right\}$ **of S in $\mathbb{R}$** *if σ is the* $\left\{\begin{array}{c}\textit{least element}\\ \textit{greatest element}\end{array}\right\}$ *of the set of all* $\left\{\begin{array}{c}\textit{upper bounds}\\ \textit{lower bounds}\end{array}\right\}$ *of S in $\mathbb{R}$.*

Remarks.

(A) If S has a supremum in $\mathbb{R}$, it is the unique supremum of S in $\mathbb{R}$. Et cetera.

(B) **Notation.** We denote the $\left\{\begin{array}{c}\textbf{supremum}\\ \textbf{infimum}\end{array}\right\}$ of S by $\left\{\begin{array}{c}\sup(S)\\ \inf(S)\end{array}\right\}$.

(C) **Terminology.** We may choose to write '*S has a supremum*' as $\sup(S)$ *exists*'. Et cetera. The situation is analogous for infimum.

9. You may write down any non-empty subset of $\mathbb{R}$ you like, and will find that if the set concerned is bounded above/below in $\mathbb{R}$, it will have a supremum/infimum in $\mathbb{R}$.

They will provide evidence for the **Least-upper-bound Axiom**, which is a fundamental property of the real number system.

Least-upper-bound Axiom for the reals (LUBA).
Let A be a non-empty subset of $\mathbb{R}$. Suppose A is bounded above in $\mathbb{R}$.
Then A has a least upper bound in $\mathbb{R}$.

The statement (LUBA) is logically equivalent to the equally 'obvious' statement:
'Greatest-lower-bound Axiom for the reals' (GLBA).
Let A be a non-empty subset of $\mathbb{R}$. Suppose A is bounded below in $\mathbb{R}$.
Then A has a greatest lower bound in $\mathbb{R}$.

(As an exercise in word games, prove that (LUBA), (GLBA) are indeed logically equivalent.)

10. Appendix: Formalization of the real number system.

Refer to the Handout *Formalization of the Real Number System as understood in School Maths*.

There we have agreed that the **real number system**, as understood in school maths, consists of:

- a set, denoted by $\mathbb{R}$, whose elements are called **real numbers**, amongst them two distinct real numbers called **zero**, **one** and denoted by 0, 1 respectively,
- the arithmetic operations $+$, $\times$, called **addition, multiplication in the reals** respectively) , and
- a subset of $\mathbb{R}$, denoted by $\mathbb{R}_{\geq 0}$, whose elements are called **non-negative real numbers**,

forming an **ordered field**, in the sense that

- the **laws of arithmetic for the reals**, namely the statements (A1)-(A11), and
- the **laws of order for the reals (compatible to the arithmetic operations)**, namely the statements (O1)-(O3),

are true statements.

Laws of arithmetic for the reals.

(A1) *For any* $a, b \in \mathbb{R}$, $a + b \in \mathbb{R}$.

(A2) *For any* $a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$.

(A3) *There exists some* $z \in \mathbb{R}$, *namely* $z = 0$, *such that for any* $a \in \mathbb{R}$, $a + z = a$ *and* $z + a = a$.

(A4) *For any* $a \in \mathbb{R}$, *there exists some* $b \in \mathbb{R}$, *called an* **additive inverse of** a, *such that* $a + b = 0$ *and* $b + a = 0$.

(A5) *For any* $a, b \in \mathbb{R}$, $a + b = b + a$.

(A6) *For any* $a, b \in \mathbb{R}$, $a \times b \in \mathbb{R}$.

(A7) *For any* $a, b, c \in \mathbb{R}$, $(a \times b) \times c = a \times (b \times c)$.

(A8) *There exists some* $u \in \mathbb{R}$, *namely* $u = 1$, *such that for any* $a \in \mathbb{R}$, $a \times u = a$ *and* $u \times a = a$.

(A9) *For any* $a \in \mathbb{R}\backslash\{0\}$, *there exists some* $b \in \mathbb{R}$, *called a* **multiplicative inverse of** a, *such that* $a \times b = 1$ *and* $b \times a = 1$.

(A10) *For any* $a, b \in \mathbb{R}$, $a \times b = b \times a$.

(A11) *For any* $a, b, c \in \mathbb{R}$, $(a + b) \times c = (a \times c) + (b \times c)$ *and* $a \times (b + c) = (a \times b) + (a \times c)$.

Subtraction '$-$' and division $\div$ in the reals are defined in terms of addition and multiplication in the reals, under the assumption of the validity of the statements (A1)-(A11).

For each $a \in \mathbb{R}$, the additive inverse of a is proved to be unique and is denoted by $-a$; when $a \neq 0$, the multiplicative inverse of a is proved to be unique and is denoted by a^{-1}.

Laws of order for the reals (compatible to the arithmetic operations).

(O1) *For any* $a, b \in \mathbb{R}_{\geq 0}$, $a + b \in \mathbb{R}_{\geq 0}$ *and* $a \times b \in \mathbb{R}_{\geq 0}$.

(O2) *For any* $a \in \mathbb{R}$, $a \in \mathbb{R}_{\geq 0}$ *or* $-a \in \mathbb{R}_{\geq 0}$.

(O3) *For any* $a \in \mathbb{R}$, *if* $a \in \mathbb{R}_{\geq 0}$ *and* $-a \in \mathbb{R}_{\geq 0}$ *then* $a = 0$.

The **usual ordering for the reals**, which is denoted by $\leq$, is defined in terms of subtraction and non-negative real numbers.

We now further agree that the real number system (as understood in school maths) satisfies **Least-upper-bound Axiom for the reals (LUBA).**

> *Let* A *be a non-empty subset of* $\mathbb{R}$. *Suppose* A *is bounded above in* $\mathbb{R}$. *Then* A *has a least upper bound in* $\mathbb{R}$.

With the inclusion of (LUBA), we have completed the formalization of the real number system as understood in school maths.

In your *mathematical analysis* course, the Least-upper-bound Axiom serves as the ultimate justification for other 'intuitively obvious' results which you have been using without questioning in *infinitesimal calculus*, such as:

- the **Bounded-Monotone Theorem for infinite sequences of real numbers**,
- the **Intermediate-Value Theorem**, and
- the **Mean-Value Theorem**.