1. **Definition**.

Let A be a set. The **power set** *of the set A is defined to be the set*

$$\{S \mid S \text{ is a subset of } A\}.$$

It is denoted by $\mathfrak{P}(A)$.

Remark. By definition, $S \in \mathfrak{P}(A)$ iff $S \subset A$.

2. **Example (1).**

$A=?$	Elements of A?	Subsets of A? Elements of $\mathfrak{P}(A)$?	$\mathfrak{P}(A)=?$
$\emptyset$	A has no element	ϕ	$\{\phi\}$
$\{0\}$	0	ϕ, $\{0\}$	$\{\phi, \{0\}\}$
$\{0,1\}$	0, 1	ϕ, $\{0\}$, $\{1\}$, $\{0,1\}$	$\{\phi, \{0\}, \{1\}, \{0,1\}\}$
$\{0,1,2\}$	0, 1, 2	ϕ, $\{0\}$, $\{1\}$, $\{2\}$, $\{0,1\}$, $\{1,2\}$, $\{0,2\}$, $\{0,1,2\}$	$\{\phi, \{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,2\}, \{0,1,2\}\}$
$\{\emptyset\}$			
$\mathfrak{P}(\emptyset)$			
$\mathfrak{P}(\{\emptyset\})$			
$\mathfrak{P}(\{\{\emptyset\}\})$			

Remarks.

(1) $\emptyset$, $\{\emptyset\}$ are different objects.

ϕ has no element.

$\{\phi\}$ is a singleton: it contains exactly one element, namely ϕ.

(2) In general, when A has exactly N elements, $\mathfrak{P}(A)$ will have exactly 2^N elements.

Proof? Apply mathematical induction.

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$\emptyset$	A has no element.	ϕ	$\{\phi\}$
$\{0\}$	0	$\phi, \{0\}$	$\{\phi, \{0\}\}$
$\{0, 1\}$	$0, 1$	$\phi, \{0\}, \{1\}, \{0,1\}$	$\{\phi, \{0\}, \{1\}, \{0,1\}\}$
$\{0, 1, 2\}$	$0, 1, 2$	$\phi, \{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,2\}, \{0,1,2\}$	$\{\phi, \{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,2\}, \{0,1,2\}\}$
$\{\emptyset\}$	ϕ	$\phi, \{\phi\}$	$\{\phi, \{\phi\}\}$
$\{\phi\} = \mathfrak{P}(\emptyset)$	ϕ	$\phi, \{\phi\}$	$\{\phi, \{\phi\}\}$
$\{\phi, \{\phi\}\} = \mathfrak{P}(\{\emptyset\})$	$\phi, \{\phi\}$	$\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}$	$\{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$
$\{\phi, \{\{\phi\}\}\} = \mathfrak{P}(\{\{\emptyset\}\})$	$\phi, \{\{\phi\}\}$	$\phi, \{\phi\}, \{\{\{\phi\}\}\}, \{\phi, \{\{\phi\}\}\}$	$\{\phi, \{\phi\}, \{\{\{\phi\}\}\}, \{\phi, \{\{\phi\}\}\}\}$

Example (2).

(a) What is $\mathfrak{P}(\mathfrak{P}(\emptyset))$ explicitly?

Ask: what is $\mathcal{P}(\phi)$? $\mathcal{P}(\phi) = \{\phi\}$.

Then ask: what is $\mathcal{P}(\mathcal{P}(\phi))$? $\mathcal{P}(\mathcal{P}(\phi)) = \mathcal{P}(\{\phi\})$

$= \{\phi, \{\phi\}\}$.

(b) What is $\mathfrak{P}(\mathfrak{P}(\{\emptyset\}))$ explicitly?

Ask: what is $\mathcal{P}(\{\phi\})$? $\mathcal{P}(\{\phi\}) = \{\phi, \{\phi\}\}$.

Then ask: what is $\mathcal{P}(\mathcal{P}(\{\phi\}))$? $\mathcal{P}(\mathcal{P}(\{\phi\})) = \mathcal{P}(\{\phi, \{\phi\}\})$

$= \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$.

(c) What is $\mathfrak{P}(\mathfrak{P}(\{\{\emptyset\}\}))$ explicitly?

Ask: what is $\mathcal{P}(\{\{\phi\}\})$?

Then ask: what is $\mathcal{P}(\mathcal{P}(\{\{\phi\}\}))$?

3. **Theorem** ($\dagger_1$).

Let A, B be sets. The following statements hold:

1. $\emptyset, A \in \mathfrak{P}(A)$. $\mathfrak{P}(A) \neq \emptyset$.
2. (a) *Suppose $A \subset B$. Then $\mathfrak{P}(A) \subset \mathfrak{P}(B)$.*
   (b) *Suppose $\mathfrak{P}(A) \subset \mathfrak{P}(B)$. Then $A \subset B$.*
   (c) *$A \subset B$ iff $\mathfrak{P}(A) \subset \mathfrak{P}(B)$.*

Proof of (1).

$\emptyset \subset A$. Then $\emptyset \in \mathfrak{P}(A)$.
$A \subset A$. Then $A \in \mathfrak{P}(A)$.
So $\mathfrak{P}(A) \neq \emptyset$. $\square$

Proof of Statements (1), (2a) of Theorem ($\dagger_1$).

Proof of (2a).

Let A, B be sets. Suppose $A \subset B$.

[Try to prove: $\mathfrak{P}(A) \subset \mathfrak{P}(B)$.
What is it, really? For any S, if $S \in \mathfrak{P}(A)$ then $S \in \mathfrak{P}(B)$.]

- Pick any object S. Suppose $S \in \mathfrak{P}(A)$. [Try to deduce: $S \in \mathfrak{P}(B)$.]
  By definition, since $S \in \mathfrak{P}(A)$, we have $S \subset A$.
  By Theorem (I.3), since $S \subset A$ and $A \subset B$, we have $S \subset B$.
  Now, by definition, since $S \subset B$, we have $S \in \mathfrak{P}(B)$.

It follows that $\mathfrak{P}(A) \subset \mathfrak{P}(B)$. $\square$

Theorem ($\dagger_1$).

Let A, B be sets. The following statements hold:

1. $\emptyset, A \in \mathfrak{P}(A)$. $\mathfrak{P}(A) \neq \emptyset$.
2. (a) *Suppose $A \subset B$. Then $\mathfrak{P}(A) \subset \mathfrak{P}(B)$.*
   (b) *Suppose $\mathfrak{P}(A) \subset \mathfrak{P}(B)$. Then $A \subset B$.*
   (c) *$A \subset B$ iff $\mathfrak{P}(A) \subset \mathfrak{P}(B)$.*

Proof of Statements (2b), (2c) of Theorem ($\dagger_1$).

Proof of (2b).

Let A, B be sets. Suppose $\mathfrak{P}(A) \subset \mathfrak{P}(B)$.

[Try to prove: $A \subset B$.
Observe this is same as: $A \in \mathfrak{P}(B)$.]

By (1), we have $A \in \mathfrak{P}(A)$.

Since $A \in \mathfrak{P}(A)$ and $\mathfrak{P}(A) \subset \mathfrak{P}(B)$, we have $A \in \mathfrak{P}(B)$ [by the definition of subsets].

Therefore, by definition, we have $A \subset B$. $\square$

Proof of (2c). The result follows from (2a), (2b) immediately. $\square$

Theorem ($\dagger_1$).

Let A, B be sets. The following statements hold:

1. $\emptyset, A \in \mathfrak{P}(A)$. $\mathfrak{P}(A) \neq \emptyset$.
2. (a) ... (b) ... (c) $A \subset B$ *iff* $\mathfrak{P}(A) \subset \mathfrak{P}(B)$.
3. $\mathfrak{P}(A \cap B) = \mathfrak{P}(A) \cap \mathfrak{P}(B)$.

(3a) $\mathfrak{P}(A \cap B) \subset \mathfrak{P}(A) \cap \mathfrak{P}(B)$.

This is:

For any S, if $S \in \mathfrak{P}(A \cap B)$ then $S \in \mathfrak{P}(A) \cap \mathfrak{P}(B)$.

(3b) $\mathfrak{P}(A) \cap \mathfrak{P}(B) \subset \mathfrak{P}(A \cap B)$.

This is:

For any T, if $T \in \mathfrak{P}(A) \cap \mathfrak{P}(B)$ then $T \in \mathfrak{P}(A \cap B)$.

Proof of Statement (3) of Theorem ($\dagger_1$).

Let A, B be sets. [What to prove? Some set equality. So?]

- [Proof of (3a). (We can make use of previously proved results.)]

  Since $A \cap B \subset A$, we have $\mathfrak{P}(A \cap B) \subset \mathfrak{P}(A)$ by (2a).

  Since $A \cap B \subset B$, we have $\mathfrak{P}(A \cap B) \subset \mathfrak{P}(B)$ by (2a).

  Now, by Theorem (IV.1), $\mathfrak{P}(A \cap B) \subset \mathfrak{P}(A) \cap \mathfrak{P}(B)$.

- [Proof of (3b).]

  Pick any object T. Suppose $T \in \mathfrak{P}(A) \cap \mathfrak{P}(B)$. [Try to deduce: $T \in \mathfrak{P}(A \cap B)$.]

  We have $T \in \mathfrak{P}(A)$ and $T \in \mathfrak{P}(B)$.

  Since $T \in \mathfrak{P}(A)$, we have $T \subset A$. Since $T \in \mathfrak{P}(B)$, we have $T \subset B$.

  Now $T \subset A$ and $T \subset B$. Then, by Theorem (IV.1), $T \subset A \cap B$.

  Then, by definition, $T \in \mathfrak{P}(A \cap B)$. It follows that $\mathfrak{P}(A) \cap \mathfrak{P}(B) \subset \mathfrak{P}(A \cap B)$.

... $\square$

Theorem ($\dagger_1$).

Let A, B be sets. The following statements hold:

1. $\emptyset, A \in \mathfrak{P}(A)$. $\mathfrak{P}(A) \neq \emptyset$.
2. (a) ... (b) ... (c) $A \subset B$ *iff* $\mathfrak{P}(A) \subset \mathfrak{P}(B)$.
3. $\mathfrak{P}(A \cap B) = \mathfrak{P}(A) \cap \mathfrak{P}(B)$.
4. $\mathfrak{P}(A) \cup \mathfrak{P}(B) \subset \mathfrak{P}(A \cup B)$.

Proof of Statement (4) of Theorem ($\dagger_1$).

Let A, B be sets.

Since $A \subset A \cup B$, we have $\mathfrak{P}(A) \subset \mathfrak{P}(A \cup B)$ by (2a).

Since $B \subset A \cup B$, we have $\mathfrak{P}(B) \subset \mathfrak{P}(A \cup B)$ by (2a).

Then $\mathfrak{P}(A) \cup \mathfrak{P}(B) \subset \mathfrak{P}(A \cup B)$. (why?) $\square$

4. It is natural to ask whether Statement (‡) below is true:

(‡) *Let A, B be sets.* $\mathfrak{P}(A \cup B) = \mathfrak{P}(A) \cup \mathfrak{P}(B)$.

This is the same as asking whether Statement (‡′) below is true:

(‡′) *Let A, B be sets.* $\mathfrak{P}(A \cup B) \subset \mathfrak{P}(A) \cup \mathfrak{P}(B)$.

How to approach such a question? Ask Question (◇):

(◇) What happens to the sets A, B if the conclusion '$\mathfrak{P}(A \cup B) \subset \mathfrak{P}(A) \cup \mathfrak{P}(B)$' holds?

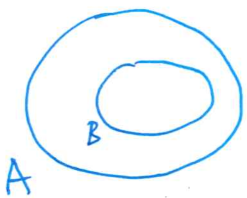
Answer for Question (◇):

Note that $A \cup B \in \mathfrak{P}(A \cup B)$.

Suppose it happens that '$\mathfrak{P}(A \cup B) \subset \mathfrak{P}(A) \cup \mathfrak{P}(B)$' holds.

Then $A \cup B \in \mathfrak{P}(A \cup B) \subset \mathfrak{P}(A) \cup \mathfrak{P}(B)$.

Therefore $\underbrace{A \cup B \in \mathfrak{P}(A)}$ or $\underbrace{A \cup B \in \mathfrak{P}(B)}$



This is same as '$A \cup B \subset A$'. In fact this is same as '$B \subset A$'.

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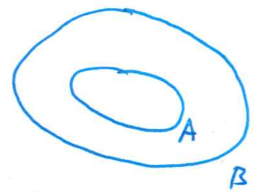


This is something 'non-trivial'.

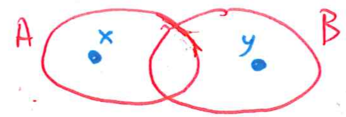
5. In fact a necessary condition for

$$\text{`}\mathfrak{P}(A \cup B) \subset \mathfrak{P}(A) \cup \mathfrak{P}(B)\text{'}$$

is given by '$B \subset A$ or $A \subset B$'.

It turns out that 'this necessary condition is also sufficient'.

If this happens



then there is no chance for '$\mathfrak{P}(A \cup B) \subset \mathfrak{P}(A) \cup \mathfrak{P}(B)$' to hold.

Theorem ($\dagger_2$). *Let A, B be sets. The following statements hold:*

1. Suppose $\mathfrak{P}(A \cup B) \subset \mathfrak{P}(A) \cup \mathfrak{P}(B)$. Then ($B \subset A$ or $A \subset B$).
2. Suppose ($B \subset A$ or $A \subset B$). Then $\mathfrak{P}(A \cup B) \subset \mathfrak{P}(A) \cup \mathfrak{P}(B)$.
3. $\mathfrak{P}(A \cup B) \subset \mathfrak{P}(A) \cup \mathfrak{P}(B)$ iff ($B \subset A$ or $A \subset B$).

6. **Theorem** ($\star$). *Let M be a set.*

   1. *$\mathfrak{P}(M)$ is partially ordered by the subset relation.*
   2. *$\mathfrak{P}(M)$ together with the set operations intersection, union, and complement in M constitutes a boolean algebra.*
   3. *$\mathfrak{P}(M)$ form an abelian group with symmetric difference in M as group operation.*