MATH1050 Existence, uniqueness, and existence-and-uniqueness

1. **Existence versus uniqueness.**

   A statement of the form

   ‘$(\exists x)Q(x)$’,
   ‘$(\exists x)(\exists y)Q(x,y)$’,
   ...,
   ‘$(\exists x)(\exists y)\cdots Q(x,y,\cdots)$’

   is called an **existence statement**.

   In ordinary language, we write: ‘*there is at least one x for which $Q(x)$ is true*’, ‘*there is at least a pair of x, y for which $Q(x,y)$ is true*’, *et cetera*.

   A statement of the form

   ‘$(\forall x)(\forall x')[(Q(x) \wedge Q(x')) \rightarrow (x = x')]$’,
   ‘$(\forall x)(\forall y)(\forall x')(\forall y')[(Q(x,y) \wedge Q(x',y')) \rightarrow [(x = x') \wedge (y = y')]]$’,
   ...,
   ‘$(\forall x)(\forall y)(\forall x')(\forall y')\cdots[(Q(x,y,\cdots) \wedge Q(x',y',\cdots)) \rightarrow [(x = x') \wedge (y = y') \wedge \cdots]]$’

   is called a **uniqueness statement**.

   In ordinary language, we write: ‘*there is at most one x for which $Q(x)$ is true*’, ‘*there is at most a pair of x, y for which $Q(x,y)$ is true*’, *et cetera*.

   Remark. The two statements

   - ‘*there is at least one x for which $Q(x)$ is true*’,
   - ‘*there is at most one x for which $Q(x)$ is true*’,

   may look ‘similar’ at first sight (differing from each other by only one word), but they are very different, as we can see from their respective ‘formal formulations’

   - ‘$(\exists x)Q(x)$’,
   - ‘$(\forall x)(\forall x')[(Q(x) \wedge Q(x')) \rightarrow (x = x')]$’.

   Given a ‘concrete’ $Q(x)$, it can happen that both are true or that both are false, or that one is true and the other is false. Neither implies the other.

2. **Examples of existence statements.**

   The parts of the statements below highlighted in boldface constitute ‘existence statements’.

   (a) ‘Defining condition’ in the definition of divisibility.

   Let $m, n \in \mathbb{Z}$. m is said to be divisible by n if **there exists some** $k \in \mathbb{Z}$ **such that** $m = kn$.

   Remark.

   (A) What does ‘there exists some $k \in \mathbb{N}$ such that $m = kn$’ say, really?

   (1) The already present m, n ‘generates’ this k, whose exact value depends on the values of m, n.

   (2) $k \in \mathbb{Z}$.

   (3) m, n, k are related by $m = kn$.

   (B) When $m = n = 0$, there are more than one such k. But this does not matter.

   (b) Mean-Value Theorem.

   Let $a, b \in \mathbb{R}$. Suppose $a < b$. Let $f : [a,b] \longrightarrow \mathbb{R}$ be a function. Suppose f is continuous on $[a,b]$ and f is differentiable on (a,b). Then **there exists some** $\gamma \in (a,b)$ **such that** $f(b) - f(a) = (b-a)f'(\gamma)$.

   Remark.

   (A) What does ‘there exists some $\gamma \in (a,b)$ such that $f(b) - f(a) = (b-a)f'(\gamma)$’ say, really?

   (1) The already present a, b, f ‘generates’ this γ, whose exact value depends on the values of a, b and the ‘formula’ of f.

   (2) $\gamma \in (a,b)$.

   (3) a, b, f, γ are related by $f(b) - f(a) = (b-a)f'(\gamma)$.

   (B) It can happen that there are more than one such γ. But this does not matter.

   (c) Well-Ordering Principle for integers.

   Let S be a non-empty subset of $\mathbb{N}$. S **has at least one least element**.

(d) Archimedean Principle.

Let $x, u \in (0, +\infty)$. **There exists some $N \in \mathbb{N}$ such that** $x \leq Nu$.

(e) Least-upper-bound Principle.

Let S be a non-empty subset of $\mathbb{R}$. Suppose S has an upper bound in $\mathbb{R}$. Then S **has a least upper bound**.

Remark. In fact, the assumption 'S has an upper bound in $\mathbb{R}$' is also an existence statement.

(f) *Let S be a spanning set for the vector space $\mathbb{R}^n$ over $\mathbb{R}$. Suppose $\mathbf{x} \in \mathbb{R}^n$. Then* **there exist some k vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in S$, some k real numbers $c_1, c_2, \cdots, c_k$ such that $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$.**

3. **Examples of uniqueness statements**.

The parts of the statements below highlighted in boldface constitute 'uniqueness statements'.

(a) *Let S be a subset of $\mathbb{R}$.* S **has at most one least element.**

Remark.

(A) What does 'S has at most one least element' say, really?

- If λ, λ' are two least elements of S, then $\lambda = \lambda'$.

(B) What to verify in order to prove that S has at most one least element?

- Let S be a subset of $\mathbb{R}$. Let $\lambda, \lambda' \in S$. Suppose each of λ, λ' are least elements of S. Then $\lambda = \lambda'$.

(C) Nothing is said on whether or not such a set S has a least element at all. But this does not matter.

(b) **There is at most one line in the Argand plane $\mathbb{C}$ passing through all of the points $0, 1, i$.**

Remark.

(A) What does 'there is at most one line in the Argand plane $\mathbb{C}$ passing through all of the points $0, 1, i$' say, really?

- If ℓ, ℓ' are lines in the Argand plane $\mathbb{C}$, each passing through all of $0, 1, i$, then $\ell = \ell'$.

(B) What to verify in order to prove that there is at most one line in the Argand plane $\mathbb{C}$ passing through all of the points $0, 1, i$?

- *Let ℓ, ℓ' be lines in the Argand plane $\mathbb{C}$. Suppose ℓ passes through all of $0, 1, i$. Also suppose ℓ' passes through all of $0, 1, i$. Then $\ell = \ell'$.*

(C) There is, of course, no line passing through all of the points $0, 1, i$. But this does not matter.

(c) *Let $A, B, C, D \in \mathbb{R}^2$.* **There is at most one circle passing through** A, B, C, D.

(d) *Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{R}$.* $\{a_n\}_{n=0}^{\infty}$ **has at most one limit in $\mathbb{R}$.**

(e) *Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ be k linearly independent vectors in $\mathbb{R}^n$ over $\mathbb{R}$. Suppose $\mathbf{x} \in \mathbb{R}^n$. Then* **there is at most one collection of k real numbers $c_1, c_2, \cdots, c_k$ which satisfies $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$.**

4. **Existence-and-uniqueness**.

An **existence-and-uniqueness statement** is a statement of the form:

- $[(\exists x)(\exists y) \cdots Q(x, y, \cdots)] \quad \wedge \quad \{(\forall x)(\forall y)(\forall x')(\forall y') \cdots [(Q(x, y, \cdots) \wedge Q(x', y', \cdots)) \rightarrow [(x = x') \wedge (y = y') \wedge \cdots]]\}$

Consider the words highlighted in boldface in the statement of the 'Playfair Axiom' in Euclidean geometry below:

- Let ℓ be a line in (the Euclidean plane) $\mathbb{R}^2$ and p be a point in $\mathbb{R}^2$. Suppose p does not lie on ℓ. Then **there exists some unique line m in $\mathbb{R}^2$ such that m does not intersect ℓ and p lies on m.**

With ℓ, p being specified, the words 'there exists some unique ...' constitutes an existence-and-uniqueness statement. It is the conjunction of an existence statement and a uniqueness statement, referred to as the **existence part** and the **uniqueness part** respectively:

- 'Existence part' (as highlighted in boldface):

  (Let ℓ be a line in $\mathbb{R}^2$ and p be a point in $\mathbb{R}^2$. Suppose p does not lie on ℓ.) Then **there exists some line m in $\mathbb{R}^2$ such that m does not intersect ℓ and p lies on m.**

- 'Uniqueness part' (as highlighted in boldface):

  (Let ℓ be a line in $\mathbb{R}^2$ and p be a point in $\mathbb{R}^2$. Suppose p does not lie on ℓ.) **Let m, m' be lines in $\mathbb{R}^2$. Suppose each of m, m' does not intersect ℓ and p lies on each of m, m'. Then $m = m'$.**

The respective notions of existence and uniqueness are independent. An existence-and-uniqueness statement is true exactly when its 'existence part' is true and its 'uniqueness part' is true. Hence, to prove a statement, we should give the arguments for its 'existence part' and its 'uniqueness part' separately.

5. **Examples of existence-and-uniqueness statements.**

The parts of the statements below highlighted in boldface constitute 'existence-and-uniqueness statements'.

(a) Theorem (DAN).

Let $m, n \in \mathbb{N}$. *Suppose* $n \neq 0$. *Then* **there exist some unique** $q, r \in \mathbb{N}$ **such that** $m = qn + r$ **and** $0 \leq r < n$.

- 'Existence part'.

  Let $m, n \in \mathbb{N}$. *Suppose* $n \neq 0$. *Then* **there exist some** $q, r \in \mathbb{N}$ **such that** $m = qn+r$ **and** $0 \leq r < n$.

- 'Uniqueness part'.

  Let $m, n \in \mathbb{N}$. *Suppose* $n \neq 0$. **Suppose** q, r, q', r' **are natural numbers satisfying** $m = qn + r$, $m = q'n + r'$ **and** $0 \leq r < n$, $0 \leq r' < n$. **Then** $q = q'$ **and** $r = r'$**.**

(b) *Let* p, q *be distinct points in* $\mathbb{R}^2$. **There exists some unique line** ℓ **in** $\mathbb{R}^2$ **such that** p, q **lie on** ℓ.

- 'Existence part'.

  Let p, q *be distinct points in* $\mathbb{R}^2$. **There exist some line** ℓ **in** $\mathbb{R}^2$ **such that** p, q **lie on** ℓ**.**

- 'Uniqueness part'.

  Let p, q *be distinct points in* $\mathbb{R}^2$. **Suppose** ℓ, ℓ' **are lines in** $\mathbb{R}^2$**. Suppose both of** p, q **lie on each of** ℓ, ℓ'**. Then** $\ell = \ell'$.

(c) *Let* ℓ *be a line in* $\mathbb{R}^2$ *and* p *be a point in* $\mathbb{R}^2$. *Suppose* p *does not lie on* ℓ. *Then* **there exists some unique line** n **in** $\mathbb{R}^2$ **such that** n **is perpendicular to** ℓ **and** p **lies on** n.

- 'Existence part'.

  Let ℓ *be a line in* $\mathbb{R}^2$ *and* p *be a point in* $\mathbb{R}^2$. *Suppose* p *does not lie on* ℓ. *Then* **there exists some line** n **in** $\mathbb{R}^2$ **such that** n **is perpendicular to** ℓ **and** p **lies on** n**.**

- 'Uniqueness part'.

  Let ℓ *be a line in* $\mathbb{R}^2$ *and* p *be a point in* $\mathbb{R}^2$. *Suppose* p *does not lie on* ℓ. **Suppose** n, n' **are lines in** $\mathbb{R}^2$**, each being perpendicular to** ℓ **and containing** p**. Then** $n = n'$**.**

(d) *Let* A *be an* $n \times n$*-matrix with real entries, and* $\mathbf{b} \in \mathbb{R}^n$. *Suppose* $\det(A) \neq 0$. *Then* **the equation** $A\mathbf{x} = \mathbf{b}$ **(with unknown vector x in** $\mathbb{R}^n$**) has a unique solution in** $\mathbb{R}^n$.

- 'Existence part'.

  Let A *be an* $n \times n$*-matrix with real entries, and* $\mathbf{b} \in \mathbb{R}^n$. *Suppose* $\det(A) \neq 0$. *Then* **the equation** $A\mathbf{x} = \mathbf{b}$ **has a solution in** $\mathbb{R}^n$**.**

- 'Uniqueness part'.

  Let A *be an* $n \times n$*-matrix with real entries, and* $\mathbf{b} \in \mathbb{R}^n$. *Suppose* $\det(A) \neq 0$. **Suppose** $\mathbf{u}, \mathbf{v}$ **are both solutions of the equation** $A\mathbf{x} = \mathbf{b}$ **in** $\mathbb{R}^n$**. Then** $\mathbf{u} = \mathbf{v}$**.**

(e) *Let* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be* n *vectors which form a base for the vector space* $\mathbb{R}^n$ *over* $\mathbb{R}$. *Suppose* $\mathbf{x} \in \mathbb{R}^n$. *Then* **there exist some unique** $c_1, c_2, \cdots, c_n \in \mathbb{R}$ **such that** $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n$**.**

- 'Existence part'.

  Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be* n *vectors which form a base for the vector space* $\mathbb{R}^n$ *over* $\mathbb{R}$. *Suppose* $\mathbf{x} \in \mathbb{R}^n$. *Then* **there exist some** $c_1, c_2, \cdots, c_n \in \mathbb{R}$ **such that** $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n$**.**

- 'Uniqueness part'.

  Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be* n *vectors which form a base for the vector space* $\mathbb{R}^n$ *over* $\mathbb{R}$. *Suppose* $\mathbf{x} \in \mathbb{R}^n$. **Suppose** $c_1, c_2, \cdots, c_n, c_1', c_2', \cdots, c_n' \in \mathbb{R}$**. Suppose** $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n$**. Also suppose** $\mathbf{x} = c_1'\mathbf{u}_1 + c_2'\mathbf{u}_2 + \cdots + c_n'\mathbf{u}_n$**. Then** $c_1 = c_1'$**,** $c_2 = c_2'$**, ...,** $c_n = c_n'$**.**

(f) *Let* $a, b, x_0, y_0 \in \mathbb{R}$. *Suppose* $a < x_0 < b$. *Let* $f : (a, b) \longrightarrow \mathbb{R}$ *be a function. Suppose* f *is continuous on* (a, b). *Then* **there exists some unique continuously differentiable function** $F : (a, b) \longrightarrow \mathbb{R}$ **such that** $F(x_0) = y_0$ **and for any** $x \in (a, b)$**,** $F'(x) = f(x)$**.**

- 'Existence part'.

  Let $a, b, x_0, y_0 \in \mathbb{R}$. *Suppose* $a < x_0 < b$. *Let* $f : (a, b) \longrightarrow \mathbb{R}$ *be a function. Suppose* f *is continuous on* (a, b). *Then* **there exists some continuously differentiable function** $F : (a, b) \longrightarrow \mathbb{R}$ **such that** $F(x_0) = y_0$ **and for any** $x \in (a, b)$**,** $F'(x) = f(x)$**.**

- 'Uniqueness part'.

  Let $a, b, x_0, y_0 \in \mathbb{R}$. *Suppose* $a < x_0 < b$. *Let* $f : (a, b) \longrightarrow \mathbb{R}$ *be a function. Suppose* f *is continuous on* (a, b). **Suppose** $F, G : (a, b) \longrightarrow \mathbb{R}$ **are continuously differentiable functions. Suppose** $F(x_0) = y_0$ **and** $G(x_0) = y_0$**. Suppose that for any** $x \in (a, b)$**,** $F'(x) = f(x)$**. Suppose that for any** $x \in (a, b), G'(x) = f(x)$**. Then** F **and** G **are the same function on** (a, b).