MATH1050 Statements with several quantifiers

1. **Statements with many quantifiers**.

   When we carefully analyse the logical structure of a mathematical statement, say, S, we will most likely find that S is of the form
   $$(\mathsf{q}_x x)((\mathsf{q}_y y)(\cdots((\mathsf{q}_z z)((\mathsf{q}_w w)P(x, y, \cdots, z, w)))\cdots)),$$
   in which:

   - $P(x, y, \cdots, z, w)$ is a predicate with variables $x, y, \cdots, z, w$, and
   - each of q_x, q_y, ..., q_z, q_w stands for the universal quantifier $\forall$ or the existential quantifier $\exists$.

   Starting with the predicate $P(x, y, \cdots, z, w)$, we obtain successive predicates with fewer and fewer variables, arriving at the statement S in the end, by 'closing the variables $w, z, \cdots, y, x$ with quantifiers' one by one:

   - $P(x, y, \cdots, z, w)$,
   - $(\mathsf{q}_w w)P(x, y, \cdots, z, w)$,
   - $(\mathsf{q}_z z)((\mathsf{q}_w w)P(x, y, \cdots, z, w))$,
   - ...
   - $(\mathsf{q}_y y)(\cdots((\mathsf{q}_z z)((\mathsf{q}_w w)P(x, y, \cdots, z, w)))\cdots)$,
   - $(\mathsf{q}_x x)((\mathsf{q}_y y)(\cdots((\mathsf{q}_z z)((\mathsf{q}_w w)P(x, y, \cdots, z, w)))\cdots))$.

2. **Statements starting with two quantifiers**.

   From a predicate $Q(x, y)$ with two variables x, y, eight statements can be formed:

   (1) $(\forall x)[(\forall y)Q(x, y)]$.
   (2) $(\forall y)[(\forall x)Q(x, y)]$.
   (3) $(\exists x)[(\exists y)Q(x, y)]$.
   (4) $(\exists y)[(\exists x)Q(x, y)]$.
   (5) $(\forall x)[(\exists y)Q(x, y)]$.
   (6) $(\exists y)[(\forall x)Q(x, y)]$.
   (7) $(\exists x)[(\forall y)Q(x, y)]$.
   (8) $(\forall y)[(\exists x)Q(x, y)]$.

   We accept (1), (2) to be logically equivalent.

   Examples of (1), (2).

   (a) *For any $x > 0$, for any $y > 0$, $x + y > 0$.*
   (b) *Let $x, y \in \mathbb{Z}$. Suppose x is divisible by y and y is divisible by x. Then $|x| = |y|$.*

   We accept (3), (4) to be logically equivalent (in most situations).

   Examples of (3), (4).

   (a) *There exist some irrational numbers x, y such that $x + y$ is a rational number.*
   (b) *There exist some integers q, r such that $10000 = 333q + r$ and $0 \leq r \leq 332$.*

   Care must be taken with (5), (6), (7), (8).

3. **Statements starting with one universal quantifier and one existential quantifier**.

   Non-mathematical examples. Compare and contrast the statements in each pair $(\flat), (\sharp)$ below:

   (a) $(\flat)$ *Every student gets A in some MATH course.*
   (No big deal; everyone has his/her own 'lucky' course.)
   $(\sharp)$ *In some MATH course, every student gets A.*
   (Then you will rush to enrole in such a course.)

   (b) $(\flat)$ *In every MATH course, some student gets A.*
   (No big deal; you don't expect us to be excessively harsh.)
   $(\sharp)$ *Some student gets A in every MATH course.*
   (Then you will look for 'source' from him/her.)

   Now replace 'A' by 'F', and compare and contrast the resultant statements.

(a') ($\flat$) *Every student gets F in some MATH course.*
(Then getting F is nothing, but you still hope it happens to you no more than once.)

($\sharp$) *In some MATH course, every student gets F.*
(Then you will hope this is not a compulsory course.)

(b') ($\flat$) *In every MATH course, some student gets F.*
(Then you will work hard and pray you are not those hopefully very few ones.)

($\sharp$) *Some student gets F in every MATH course.*
(You will probably not find him/her as a classmate next year.)

When interpreting a statement with two or more quantifiers, we have to be careful with the 'relative positioning' of the quantifiers.

Mathematical Examples. Compare and contrast the statements in each pair ($\flat$), ($\sharp$) below:

(c) ($\flat$) *For any $x \in \mathbb{R}$, there exists some $y \in \mathbb{R}$ such that $x < y$.*
($\sharp$) *There exists some $y \in \mathbb{R}$ such that for any $x \in \mathbb{R}$, $x < y$.*
($\flat$) is true and ($\sharp$) is false.

(d) ($\flat$) *For any triangle x, there exists some circle y such that y passes through all three vertices of x.*
($\sharp$) *There exists some circle y such that for any triangle x, y passes through all three vertices of x.*
($\flat$) is true and ($\sharp$) is false.

(e) ($\flat$) *For any $x \in \mathbb{Z}$, there exists some $y \in \mathbb{Z}$ such that $x + y = x$.*
($\sharp$) *There exists some $y \in \mathbb{Z}$ such that for any $x \in \mathbb{Z}$, $x + y = x$.*
Both ($\flat$), ($\sharp$) are true. (($\sharp$) is useful, ($\flat$) useless.)

(f) ($\flat$) *(Let S be a non-empty subset of $\mathbb{N}$.) For any $x \in S$, there exists some $y \in S$ such that $y \leq x$.*
($\sharp$) *(Let S be a non-empty subset of $\mathbb{N}$.) There exists some $y \in S$ such that for any $x \in S$, $y \leq x$.*
Both ($\flat$), ($\sharp$) are true. (($\sharp$) is useful, ($\flat$) useless.)

In each of these examples, we have a pair of statements of the form:

($\flat$) $(\forall x)[(\exists y)Q(x, y)]$. ($\sharp$) $(\exists y)[(\forall x)Q(x, y)]$.

They are resultants of different 'sequences' in 'closing variables with quantifiers':

- How to obtain ($\flat$)? First $Q(x, y)$; next $(\exists y)Q(x, y)$; finally $(\forall x)[(\exists y)Q(x, y)]$.
- How to obtain ($\sharp$)? First $Q(x, y)$; next $(\forall x)Q(x, y)$; finally $(\exists y)[(\forall x)Q(x, y)]$.

The convention for ($\flat$) to be understood is:

- For any object x, there exists some object y_x, *depending* on what x is (as indicated by the subscript 'x' in 'y_x') such that $Q(x, y_x)$ is a true statement.

The convention for ($\sharp$) to be understood is:

- There exists some object y such that for any object x, $Q(x, y)$ is a true statement.

If (for some very good reason,) you need start with 'for any ojbect x' in a 'wordy' formulation of ($\sharp$), you must write in this way:

- For any object x, there exists some object y *independent* of the choice of x such that $Q(x, y)$ is a true statement.

Warning. Always remember these points when you read or write a statement involving both the universal quantifier and the existential quantifier:

(a) The statements ($\flat$), ($\sharp$) are different.
  - ($\sharp$) implies ($\flat$): if ($\sharp$) is true then ($\flat$) is true.
  - However, ($\flat$) does not imply ($\sharp$): when ($\flat$) is true, ($\sharp$) may be true or false.

(b) The 'relative positioning' of '$\forall x$', '$\exists y$' cannot be interchanged.
  - In ($\flat$), y 'depends' on x.

- In ($\sharp$), y does not 'depend' on x.

(c) If you are in doubt, recall some examples which help you distinguish the meanings of ($\flat$) and ($\sharp$). For example, refer to 'non-mathematical examples'.

(d) Ask yourself whether what you write is the same as what you will be understood. For instance, if what you mean is

'*for any* $x \in \mathbb{R}$, *there exists some* $y \in \mathbb{R}$ *such that* $x < y$',

do not write the statement as

'*for any* $x \in \mathbb{R}$, $x < y$ *for some* $y \in \mathbb{R}$',

or worse,

'*there exists some* $y \in \mathbb{R}$ *such that* $x < y$ *for any* $x \in \mathbb{R}$'.

4. **Negations of statements starting with two quantifiers**.

We apply the rules for negating statements with one quantifier repeatedly for statements with two quantifiers:

(a) The negation of '$(\forall x)[(\exists y)Q(x,y)]$' is '$(\exists x)[(\forall y)(\sim Q(x,y))]$'.

(b) The negation of '$(\exists y)[(\forall x)Q(x,y)]$' is '$(\forall y)[(\exists x)(\sim Q(x,y))]$'.

(c) The negation of '$(\forall x)[(\forall y)Q(x,y)]$' is '$(\exists x)[(\exists y)(\sim Q(x,y))]$'.

(d) The negation of '$(\exists x)[(\exists y)Q(x,y)]$' is '$(\forall x)[(\forall y)(\sim Q(x,y))]$'.

Examples. How to write down the negations of the statements below?

(a) *There exists some* $y \in S$ *such that for any* $x \in T$, $x < y$.

Convert the statement to be negated into a 'chain of symbols':

- $(\exists y \in S)[(\forall x \in T)(x < y)]$.

Now repeatedly apply the rules for negating statements with one quantifier:

- $\sim\{(\exists y \in S)[(\forall x \in T)(x < y)]\}$ is equivalent to $(\forall y \in S)\{\sim[(\forall x \in T)(x < y)]\}$.
The latter is logically equivalent to $(\forall y \in S)\{(\exists x \in T)[\sim(x < y)]\}$.
The latter is logically equivalent to $(\forall y \in S)[(\exists x \in T)(x \geq y)]$.

Now convert this last 'chain of symbols' into words:

- '*For any* $y \in S$, *there exists some* $x \in T$ *such that* $x \geq y$.'

This is the required negation of the given statement.

(b) *For any* $a, b \in \mathbb{Z}$, $a + b$ *is divisible by* 2.

Negation: *There exist some* $a, b \in \mathbb{Z}$ *such that* $a + b$ *is not divisible by* 2.

(c) *For any* $z \in \mathbb{C}$, *there exists some* $w \in \mathbb{R}$ *such that* $\mathsf{Re}(z + w) = \mathsf{Im}(z + w)$.

Negation: *There exists some* $z \in \mathbb{C}$ *such that for any* $w \in \mathbb{R}$, $\mathsf{Re}(z + w) \neq \mathsf{Im}(z + w)$.

(d) *There exist some* $s, t \in \mathbb{Q}$ *such that* ($s + t \in \mathbb{Z}$ *and* $st \notin \mathbb{Z}$).

Negation: *For any* $s, t \in \mathbb{Q}$, ($s + t \notin \mathbb{Z}$ *or* $st \in \mathbb{Z}$).

5. **Statements with many quantifiers**.

The principles in the discussion above can be extended to statements with three or more quantifiers.

Questions. How to read and/or write them? How to negate them?

6. **Examples from linear algebra.**

(a) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, and S be a subset of $\mathbb{R}^n$.

How to formulate '*every vector in* S *is a linear combination of* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *over* $\mathbb{R}$'?

- Formulation in words:
For any $\mathbf{x} \in S$, *there exist some* $a, b, c \in \mathbb{R}$ *such that* $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$.
- Formulation in symbols:
$(\forall \mathbf{x} \in S)[(\exists a \in \mathbb{R})(\exists b \in \mathbb{R})(\exists c \in \mathbb{R})(\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w})]$.

This statement is of the form $(\forall \mathbf{x})((\exists a)(\exists b)(\exists c)Q(\mathbf{x}, a, b, c))$.

How to formulate '*not every vector in* S *is a linear combination of* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *over* $\mathbb{R}$'?

- Formulation in symbols:
  $(\exists \mathbf{x} \in S)[(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})(\forall c \in \mathbb{R})(\mathbf{x} \neq a\mathbf{u} + b\mathbf{v} + c\mathbf{w})]$.
- Formulation in words:
  There exists some $\mathbf{x} \in S$ such that for any $a, b, c \in \mathbb{R}$, $\mathbf{x} \neq a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$.

(b) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

How to formulate '*$\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent over $\mathbb{R}$*'?

- Formulation in words:
  For any $a, b, c \in \mathbb{R}$, if $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$ then ($a = 0$ and $b = 0$ and $c = 0$).
- Formulation in symbols:
  $(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})(\forall c \in \mathbb{R})\{(a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}) \longrightarrow [(a = 0) \wedge (b = 0) \wedge (c = 0)]\}$.

This statement is of the form $(\forall a)(\forall b)(\forall c)Q(a, b, c)$.

How to formulate '*$\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent over $\mathbb{R}$*'?

- Formulation in symbols:
  $(\exists a \in \mathbb{R})(\exists b \in \mathbb{R})(\exists c \in \mathbb{R})\{(a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}) \wedge [(a \neq 0) \vee (b \neq 0) \vee (c \neq 0)]\}$.
- Formulation in words:
  There exist some $a, b, c \in \mathbb{R}$ such that ($a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$ and a, b, c are not all 0).

7. **Examples from calculus of one variable.**

(a) Let f be a real-valued function on $\mathbb{R}$, and $c \in \mathbb{R}$.

How to formulate '*f attains a relative minimum at c*'?

- Formulation in words:
  There exists some $\delta > 0$ such that for any $x \in \mathbb{R}$, (if $|x - c| < \delta$ then $f(x) \geq f(c)$).
- Formulation in symbols:
  $(\exists \delta > 0)\{(\forall x \in \mathbb{R})[(|x - c| < 0) \longrightarrow (f(x) \geq f(c))]\}$.

This statement is of the form $(\exists \delta)((\forall x)Q(\delta, x))$.

How to formulate '*f does not attain a relative minimum at c*'?

- Formulation in symbols:
  $(\forall \delta > 0)\{(\exists x \in \mathbb{R})[(|x - c| < 0) \wedge (f(x) < f(c))]\}$.
- Formulation in words:
  For any $\delta > 0$, there exists some $x \in \mathbb{R}$ such that ($|x - c| < \delta$ and $f(x) < f(c)$).

(b) Let f be a real-valued function on $\mathbb{R}$, and $c \in \mathbb{R}$.

How to formulate '*f is continuous at c*'?

- Formulation in words:
  For any $\varepsilon > 0$, there exists some $\delta > 0$ such that for any $x \in \mathbb{R}$, (if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$).
- Formulation in symbols:
  $(\forall \varepsilon > 0)\{(\exists \delta > 0)[(\forall x \in \mathbb{R})((|x - c| < 0) \longrightarrow (|f(x) - f(c)| < \varepsilon))]\}$.

This statement is of the form $(\forall \epsilon)[(\exists \delta)((\forall x)Q(\epsilon, \delta, x))]$.

How to formulate '*f is not continuous at c*'?

- Formulation in symbols:
  $(\exists \varepsilon > 0)\{(\forall \delta > 0)[(\exists x \in \mathbb{R})((|x - c| < 0) \wedge (|f(x) - f(c)| \geq \varepsilon))]\}$.
- Formulation in words:
  There exists some $\varepsilon > 0$ such that for any $\delta > 0$, there exists some $x \in \mathbb{R}$ such that ($|x - c| < \delta$ and $|f(x) - f(c)| \geq \varepsilon$).

Now you see why *mathematical analysis* is hard: even very basic notions involve a heavy presence of quantifiers.