

1. Statements with many quantifiers.

When we carefully analyse the logical structure of a mathematical statement, say, S , we will most likely find that S is of the form

$$(\mathbf{q}_x x)((\mathbf{q}_y y)(\cdots ((\mathbf{q}_z z)((\mathbf{q}_w w)P(x, y, \cdots, z, w)))) \cdots)),$$

in which:

- $P(x, y, \cdots, z, w)$ is a predicate with variables x, y, \cdots, z, w , and
- each of $\mathbf{q}_x, \mathbf{q}_y, \dots, \mathbf{q}_z, \mathbf{q}_w$ stands for the universal quantifier \forall or the existential quantifier \exists .

How to obtain S from $P(x, y, \cdots, z, w)$? 'Close the variables w, z, \cdots, y, x with quantifiers' one by one:

- $P(x, y, \cdots, z, w)$,
- $(\mathbf{q}_w w)P(x, y, \cdots, z, w)$,
- $(\mathbf{q}_z z)((\mathbf{q}_w w)P(x, y, \cdots, z, w))$,
- ...
- $(\mathbf{q}_y y)(\cdots ((\mathbf{q}_z z)((\mathbf{q}_w w)P(x, y, \cdots, z, w)))) \cdots$,
- $(\mathbf{q}_x x)((\mathbf{q}_y y)(\cdots ((\mathbf{q}_z z)((\mathbf{q}_w w)P(x, y, \cdots, z, w)))) \cdots)$.

2. Statements starting with two quantifiers.

From a predicate $Q(x, y)$ with two variables x, y , eight statements can be formed:

- | | |
|--|--|
| (1) $(\forall x)[(\forall y)Q(x, y)].$ | (5) $(\forall x)[(\exists y)Q(x, y)].$ |
| (2) $(\forall y)[(\forall x)Q(x, y)].$ | (6) $(\exists y)[(\forall x)Q(x, y)].$ |
| (3) $(\exists x)[(\exists y)Q(x, y)].$ | (7) $(\exists x)[(\forall y)Q(x, y)].$ |
| (4) $(\exists y)[(\exists x)Q(x, y)].$ | (8) $(\forall y)[(\exists x)Q(x, y)].$ |

We accept (1), (2) to be logically equivalent. Examples:

(a) For any $x > 0$, for any $y > 0$, $x + y > 0$. $\leftarrow (\forall x)[(\forall y)[((x > 0) \wedge (y > 0)) \rightarrow (x + y > 0)]]$.

(b) Let $x, y \in \mathbb{Z}$. Suppose x is divisible by y and y is divisible by x . Then $|x| = |y|$.
 $\leftarrow (\forall x)[(\forall y)[((x \in \mathbb{Z}) \wedge (y \in \mathbb{Z}) \wedge (y|x) \wedge (x|y)) \rightarrow (|x| = |y|)]]$.

We accept (3), (4) to be logically equivalent (in most situations). Examples:

(a) There exist some irrational numbers x, y such that $x + y$ is a rational number. \leftarrow

(b) There exist some integers q, r such that $10000 = 333q + r$ and $0 \leq r \leq 332$.

$\leftarrow (\exists q)[(\exists r)((q \in \mathbb{Z}) \wedge (r \in \mathbb{Z}) \wedge (10000 = 333q + r) \wedge (0 \leq r \leq 332))]$.

Care must be taken with (5), (6), (7), (8).

$\leftarrow (\exists x)[(\exists y)((x \in \mathbb{R} \setminus \mathbb{Q}) \wedge (y \in \mathbb{R} \setminus \mathbb{Q}) \wedge (x + y \in \mathbb{Q}))]$.

3. Statements starting with one universal quantifier and one existential quantifier.

Non-mathematical examples.

Compare and contrast the statements in each pair (b), (#) below:

(a)(b) *Every student gets A in some MATH course.*

(No big deal; everyone has his/her own 'lucky' course.)

(#) *In some MATH course, every student gets A.*

(Then you will rush to enrol in such a course.)

} waiting list : 999/999
CUSIS status : closed ■

(b)(b) *In every MATH course, some student gets A.*

(No big deal; you don't expect us to be excessively harsh.)

(#) *Some student gets A in every MATH course.*

(Then you will look for 'source' from him/her.)

} !

$Q(x, y)$: Student x gets A in MATH course y .

(a)(b) : $(\forall x)(\exists y) Q(x, y)$

(#) : $(\exists y)(\forall x) Q(x, y)$

(b)(b) : $(\forall y)(\exists x) Q(x, y)$

(#) : $(\exists x)(\forall y) Q(x, y)$

← Something seemingly special happens.

$Q(x, y)$: Student x gets F in MATH course y .

Now replace 'A' by 'F', and compare and contrast the resultant statements.

(a')(b) *Every student gets F in some MATH course.* $(\forall x)(\exists y)Q(x, y)$
(Then getting F is nothing, but you still hope it happens to you no more than once.)

(#) *In some MATH course, every student gets F.* $(\exists y)(\forall x)Q(x, y)$

⚠ → (Then you will hope this is not a compulsory course.)

(b')(b) *In every MATH course, some student gets F.* $(\forall y)(\exists x)Q(x, y)$
(Then you will work hard and pray you are not those hopefully very few ones.)

(#) *Some student gets F in every MATH course.* $(\exists x)(\forall y)Q(x, y)$

⚠ → (You will probably not find him/her as a classmate next year.)

'Moral of the story': Be careful with the 'relative positioning' of the universal and existential quantifiers.

Mathematical Examples.

Compare and contrast the statements in each pair (b), (#) below:

$$Q(x, y) : x < y.$$

- (c)(b) For any $x \in \mathbb{R}$, there exists some $y \in \mathbb{R}$ such that $x < y$. (b) : $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R}) Q(x, y)$.
- (#) There exists some $y \in \mathbb{R}$ such that for any $x \in \mathbb{R}$, $x < y$. (#) : $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R}) Q(x, y)$.

(b) is true. Justification:

- Pick any $x \in \mathbb{R}$. Take $y = x + 1$. Then $y \in \mathbb{R}$ and $x < x + 1 = y$.

(#) is false. Justification:

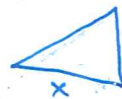
- Suppose there existed some $y \in \mathbb{R}$ such that for any $x \in \mathbb{R}$, $x < y$. Then, since $y \in \mathbb{R}$, we would have $y < y$. Contradiction arises.

- (d)(b) For any triangle x , there exists some circle y such that y passes through all three vertices of x .

- (#) There exists some circle y such that for any triangle x , y passes through all three vertices of x .

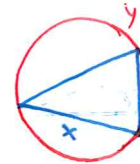
(b) is true. [Check what you learnt about circumcircles for triangles:]

Given :



...

Then :



← What y exactly is depends on what x is.

(#) is false. [Is there any circle that could simultaneously pass through all six vertices of these two triangles?]



$$Q(x, y) : x + y = x.$$

- (e)(b) For any $x \in \mathbb{Z}$, there exists some $y \in \mathbb{Z}$ such that $x + y = x$. (b) : $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})Q(x, y)$.
- (#) There exists some $y \in \mathbb{Z}$ such that for any $x \in \mathbb{Z}$, $x + y = x$. (#) : $(\exists y \in \mathbb{Z})(\forall x \in \mathbb{Z})Q(x, y)$.

(b) is true. Justification:

• Pick any $x \in \mathbb{Z}$. Take $y = 0$. Then $y \in \mathbb{Z}$ and $x + y = x + 0 = x$.

But (b) is not useful.

(#) is true and useful: it pinpoints the special nature of the integer 0:
Regardless of the value of $x \in \mathbb{Z}$, we have $x + 0 = x$.

- (f)(b) (Let S be a non-empty subset of \mathbb{N} .) For any $x \in S$, there exists some $y \in S$ such that $y \leq x$.

- (#) (Let S be a non-empty subset of \mathbb{N} .) There exists some $y \in S$ such that for any $x \in S$, $y \leq x$.

(b) is true. Justification:

• Pick any $x \in S$. Take $y = x$. Then $y \in S$ and $y = x \leq x$.

But (b) is not useful.

(#) is (believed to be) true.

It is the Well-ordering Principle for Integers.

In each of these examples, we have a pair of statements of the form:

$$(b) (\forall x)[(\exists y)Q(x, y)]. \qquad (\#) (\exists y)[(\forall x)Q(x, y)].$$

They are resultants of different ‘sequences’ in ‘closing variables with quantifiers’:

- How to obtain (b)? First $Q(x, y)$; next $(\exists y)Q(x, y)$; finally $(\forall x)[(\exists y)Q(x, y)]$.
- How to obtain (#)? First $Q(x, y)$; next $(\forall x)Q(x, y)$; finally $(\exists y)[(\forall x)Q(x, y)]$.

The convention for (b) to be understood is:

- For any object x , there exists some object y_x , *depending* on what x is (as indicated by the subscript ‘ x ’ in ‘ y_x ’) such that $Q(x, y_x)$ is a true statement.

The convention for (#) to be understood is:

- There exists some object y such that for any object x , $Q(x, y)$ is a true statement.

If (for some very good reason,) you need start with ‘for any object x ’ in a ‘wordy’ formulation of (#), you must write in this way:

- For any object x , there exists some object y *independent* of the choice of x such that $Q(x, y)$ is a true statement.

Warning. Always remember these points when you read or write a statement involving both the universal quantifier and the existential quantifier:

(a) The statements (b), (#) are different.

- (#) implies (b): if (#) is true then (b) is true.
- However, (b) does not imply (#): when (b) is true, (#) may be true or false.

(b) The 'relative positioning' of ' $\forall x$ ', ' $\exists y$ ' cannot be interchanged.

- In (b), y 'depends' on x .
- In (#), y does not 'depend' on x .

(c) If you are in doubt, recall some examples which help you distinguish the meanings of (b) and (#). For instance, refer to 'non-mathematical examples'.

(d) Ask yourself whether what you write is the same as what you will be understood.

For instance, if what you mean is

'for any $x \in \mathbb{R}$, there exists some $y \in \mathbb{R}$ such that $x < y$ ', $\leftarrow (\forall x \in \mathbb{R})((\exists y \in \mathbb{R})(x < y))$

do not write the statement as

'for any $x \in \mathbb{R}$, $x < y$ for some $y \in \mathbb{R}$ ', \leftarrow Ambiguity: Is it '(for any $x \in \mathbb{R}$, $x < y$) for some $y \in \mathbb{R}$ ' or 'for any $x \in \mathbb{R}$, ($x < y$ for some $y \in \mathbb{R}$)'?

or worse,

'there exists some $y \in \mathbb{R}$ such that $x < y$ for any $x \in \mathbb{R}$ '.

\leftarrow This is to be understood as: $(\exists y \in \mathbb{R})((\forall x \in \mathbb{R})(x < y))$

4. Negations of statements starting with two quantifiers.

We apply the rules for negating statements with one quantifier repeatedly for statements with two quantifiers:

(a) The negation of $'(\forall x)[(\exists y)Q(x, y)]'$ is

$$'(\exists x)[(\forall y)(\sim Q(x, y))]'.$$

(b) The negation of $'(\exists y)[(\forall x)Q(x, y)]'$

$$\text{is } '(\forall y)[(\exists x)(\sim Q(x, y))]'.$$

(c) The negation of $'(\forall x)[(\forall y)Q(x, y)]'$ is

$$'(\exists x)[(\exists y)(\sim Q(x, y))]'.$$

(d) The negation of $'(\exists x)[(\exists y)Q(x, y)]'$ is

$$'(\forall x)[(\forall y)(\sim Q(x, y))]'.$$

Examples. How to write down the negations of the statements below?

(a) There exists some $y \in S$ such that for any $x \in T$, $x < y$. — (*)

Convert the statement to be negated into a 'chain of symbols':

(*) reads: $(\exists y \in S) ((\forall x \in T) (x < y))$.

Now repeatedly apply the rules for negating statements with one quantifier:

Negation of (*)? • $\sim [(\exists y \in S) ((\forall x \in T) (x < y))]$
 • $(\forall y \in S) [\sim ((\forall x \in T) (x < y))]$
 • $(\forall y \in S) ((\exists x \in T) [\sim (x < y)])$

$\sim (x < y)$
 is the same as
 $x \geq y$

Now convert this last 'chain of symbols' into words:

For any $y \in S$, there exists some $x \in T$ such that $x \geq y$.

(b) For any $a, b \in \mathbb{Z}$, $a + b$ is divisible by 2. — (*)

(*) reads: $(\forall a \in \mathbb{Z}) ((\forall b \in \mathbb{Z}) (a + b \text{ is divisible by } 2))$.

Negation of (*)? • $\sim [(\forall a \in \mathbb{Z}) ((\forall b \in \mathbb{Z}) (a + b \text{ is divisible by } 2))]$
 • $(\exists a \in \mathbb{Z}) [\sim ((\forall b \in \mathbb{Z}) (a + b \text{ is divisible by } 2))]$
 • $(\exists a \in \mathbb{Z}) ((\exists b \in \mathbb{Z}) [\sim (a + b \text{ is divisible by } 2)])$.

In words, the negation of (*) reads: there exist some $a, b \in \mathbb{Z}$ such that $a + b$ is not divisible by 2.

(c) For any $z \in \mathbb{C}$, there exists some $w \in \mathbb{R}$ such that $\operatorname{Re}(z+w) = \operatorname{Im}(z+w)$. ——— (*)

(*) reads: $(\forall z \in \mathbb{C}) ((\exists w \in \mathbb{R}) (\operatorname{Re}(z+w) = \operatorname{Im}(z+w)))$

Negation of (*)? $\cdot \sim [(\forall z \in \mathbb{C}) ((\exists w \in \mathbb{R}) (\operatorname{Re}(z+w) = \operatorname{Im}(z+w)))]$

$\cdot (\exists z \in \mathbb{C}) [\sim ((\exists w \in \mathbb{R}) (\operatorname{Re}(z+w) = \operatorname{Im}(z+w)))]$

$\cdot (\exists z \in \mathbb{C}) ((\forall w \in \mathbb{R}) [\sim (\operatorname{Re}(z+w) = \operatorname{Im}(z+w))])$

\uparrow ' $\sim (\operatorname{Re}(z+w) = \operatorname{Im}(z+w))$ '
is the same as ' $\operatorname{Re}(z+w) \neq \operatorname{Im}(z+w)$ '.

In words, the negation of (*) reads: There exists some $z \in \mathbb{C}$ such that for any $w \in \mathbb{R}$, $\operatorname{Re}(z+w) \neq \operatorname{Im}(z+w)$.

(d) There exist some $s, t \in \mathbb{Q}$ such that $(s+t \in \mathbb{Z}$ and $st \notin \mathbb{Z})$. ——— (*)

(*) reads: $(\exists s \in \mathbb{Q}) ((\exists t \in \mathbb{Q}) (s+t \in \mathbb{Z} \text{ and } st \notin \mathbb{Z}))$.

Negation of (*)? $\cdot \sim [(\exists s \in \mathbb{Q}) ((\exists t \in \mathbb{Q}) (s+t \in \mathbb{Z} \text{ and } st \notin \mathbb{Z}))]$

$\cdot (\forall s \in \mathbb{Q}) [\sim ((\exists t \in \mathbb{Q}) (s+t \in \mathbb{Z} \text{ and } st \notin \mathbb{Z}))]$

$\cdot (\forall s \in \mathbb{Q}) ((\forall t \in \mathbb{Q}) [\sim (s+t \in \mathbb{Z} \text{ and } st \notin \mathbb{Z})])$

\uparrow ' $\sim (s+t \in \mathbb{Z} \text{ and } st \notin \mathbb{Z})$ ' is the same as ' $s+t \notin \mathbb{Z}$ or $st \in \mathbb{Z}$ '.

In words, the negation of (*) reads: For any $s, t \in \mathbb{Q}$, $(s+t \notin \mathbb{Z}$ or $st \in \mathbb{Z})$.

5. Statements with many quantifiers.

The principles in the discussion above can be extended to statements with three or more quantifiers.

Questions. How to read and/or write them? How to negate them?

6. Examples from linear algebra.

(a) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, and S be a subset of \mathbb{R}^n .

How to formulate 'every vector in S is a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ over \mathbb{R} '?

- Formulation in words:

(*) reads: For any $x \in S$, there exist some $a, b, c \in \mathbb{R}$ such that $x = au + bv + cw$.

- Formulation in symbols:

(*) reads: $(\forall x \in S) [(\exists a \in \mathbb{R})(\exists b \in \mathbb{R})(\exists c \in \mathbb{R}) (x = au + bv + cw)]$

How to formulate 'not every vector in S is a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ over \mathbb{R} '?

Negation of (*)

- Formulation in symbols:

Negation of (*) reads: $(\exists x \in S) [(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})(\forall c \in \mathbb{R}) (x \neq au + bv + cw)]$

- Formulation in words:

Negation of (*) reads: There exists some $x \in S$ such that for any $a, b, c \in \mathbb{R}$, $x \neq au + bv + cw$.

(b) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

How to formulate ' $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent over \mathbb{R} '?

Recall:
' $H \rightarrow K$ ' is the same
as ' $(\sim H) \vee K$ '.

• Formulation in words:

(*) reads: For any $a, b, c \in \mathbb{R}$, if $\underbrace{au + bv + cw = 0}_H$ then $\underbrace{(a=0 \text{ and } b=0 \text{ and } c=0)}_K$.
Equivalent formulation:

(*) reads: For any $a, b, c \in \mathbb{R}$, $\underbrace{[au + bv + cw \neq 0]}_{(\sim H)} \vee \underbrace{[a=0 \text{ and } b=0 \text{ and } c=0]}_K$.

• Formulation in symbols:

(*) reads: $(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})(\forall c \in \mathbb{R}) \{ (au + bv + cw = 0) \rightarrow [(a=0) \wedge (b=0) \wedge (c=0)] \}$

(*) reads: $(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})(\forall c \in \mathbb{R}) \{ (au + bv + cw \neq 0) \vee [(a=0) \wedge (b=0) \wedge (c=0)] \}$

How to formulate ' $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent over \mathbb{R} '?

• Formulation in symbols:

Negation of (*) reads: $(\exists a \in \mathbb{R})(\exists b \in \mathbb{R})(\exists c \in \mathbb{R}) \{ (au + bv + cw = 0) \wedge [(a \neq 0) \vee (b \neq 0) \vee (c \neq 0)] \}$

• Formulation in words:

Negation of (*) reads: There exist some $a, b, c \in \mathbb{R}$ such that $\underbrace{[au + bv + cw = 0]}_H \wedge \underbrace{[a \neq 0 \text{ or } b \neq 0 \text{ or } c \neq 0]}_{(\sim K)}$.

7. Examples from calculus of one variable.

(a) Let f be a real-valued function on \mathbb{R} , and $c \in \mathbb{R}$.

How to formulate ' f attains a relative minimum at c '?

• Formulation in words:

(*) reads: There exists some $\delta > 0$ such that
for any $x \in \mathbb{R}$, (if $|x-c| < \delta$ then $f(x) \geq f(c)$).

[Equivalent formulation: There exists some $\delta > 0$
such that for any $x \in \mathbb{R}$, $(|x-c| \geq \delta \vee f(x) \geq f(c))$.]

• Formulation in symbols:

(*) reads: $(\exists \delta > 0) \{ (\forall x \in \mathbb{R}) [(|x-c| < \delta) \rightarrow (f(x) \geq f(c))] \}$

Equivalent formulation: $(\exists \delta > 0) \{ (\forall x \in \mathbb{R}) [(|x-c| \geq \delta) \vee (f(x) \geq f(c))] \}$

How to formulate ' f does not attain a relative minimum at c '?

• Formulation in symbols:

Negation
of (*) reads:

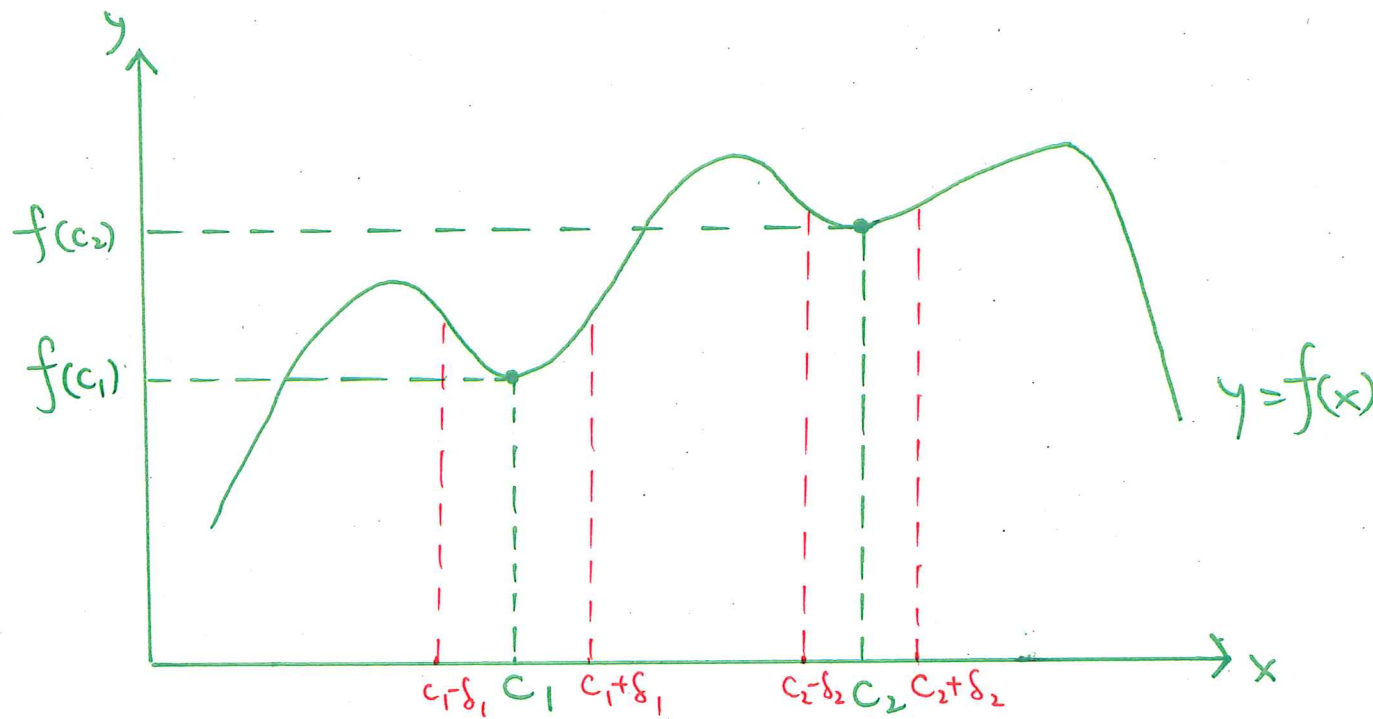
$(\forall \delta > 0) \{ (\exists x \in \mathbb{R}) [(|x-c| < \delta) \wedge (f(x) < f(c))] \}$

• Formulation in words:

Negation
of (*) reads:

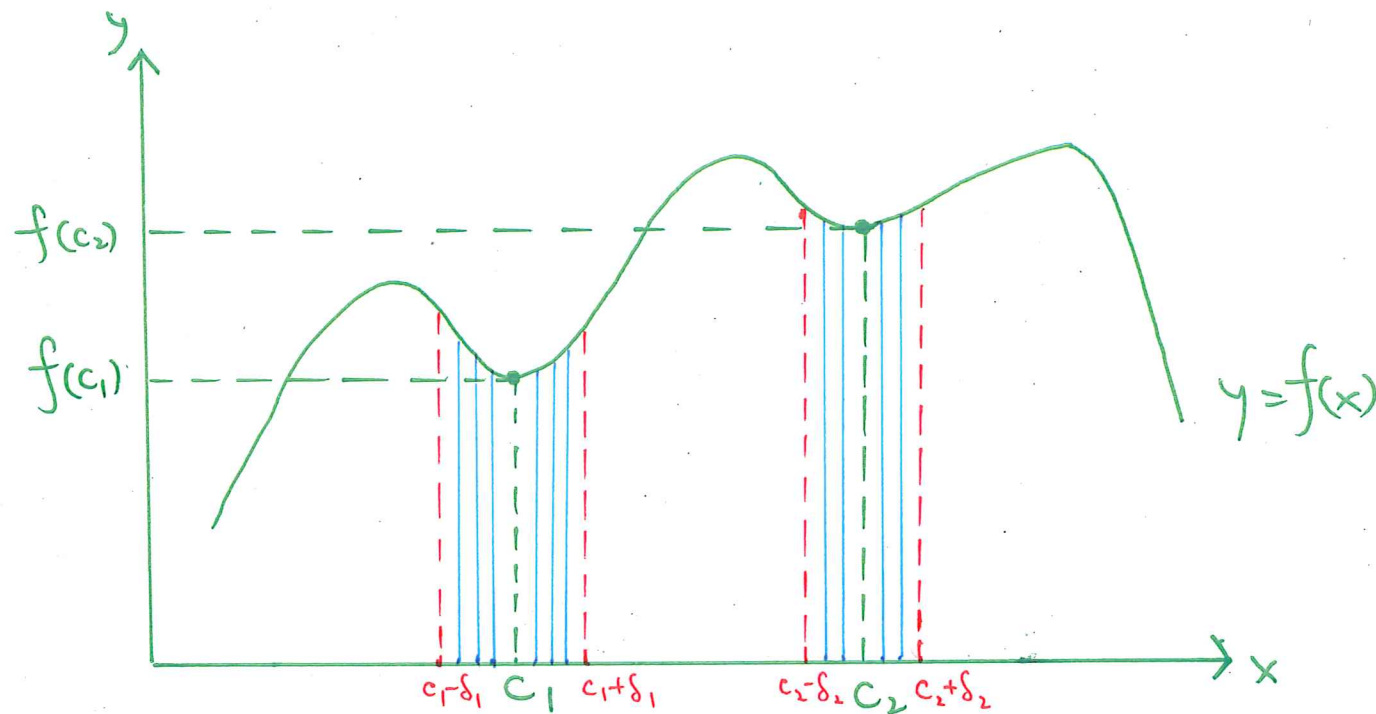
For any $\delta > 0$, there exists some $x \in \mathbb{R}$ such that
 $(|x-c| < \delta$ and $f(x) < f(c))$.

Let f be a function defined on \mathbb{R} , and $c \in \mathbb{R}$.
 f attains a relative minimum at $c \iff \exists \delta > 0$ such that for any $x \in \mathbb{R}$, if $|x - c| < \delta$ then $f(x) \geq f(c)$.



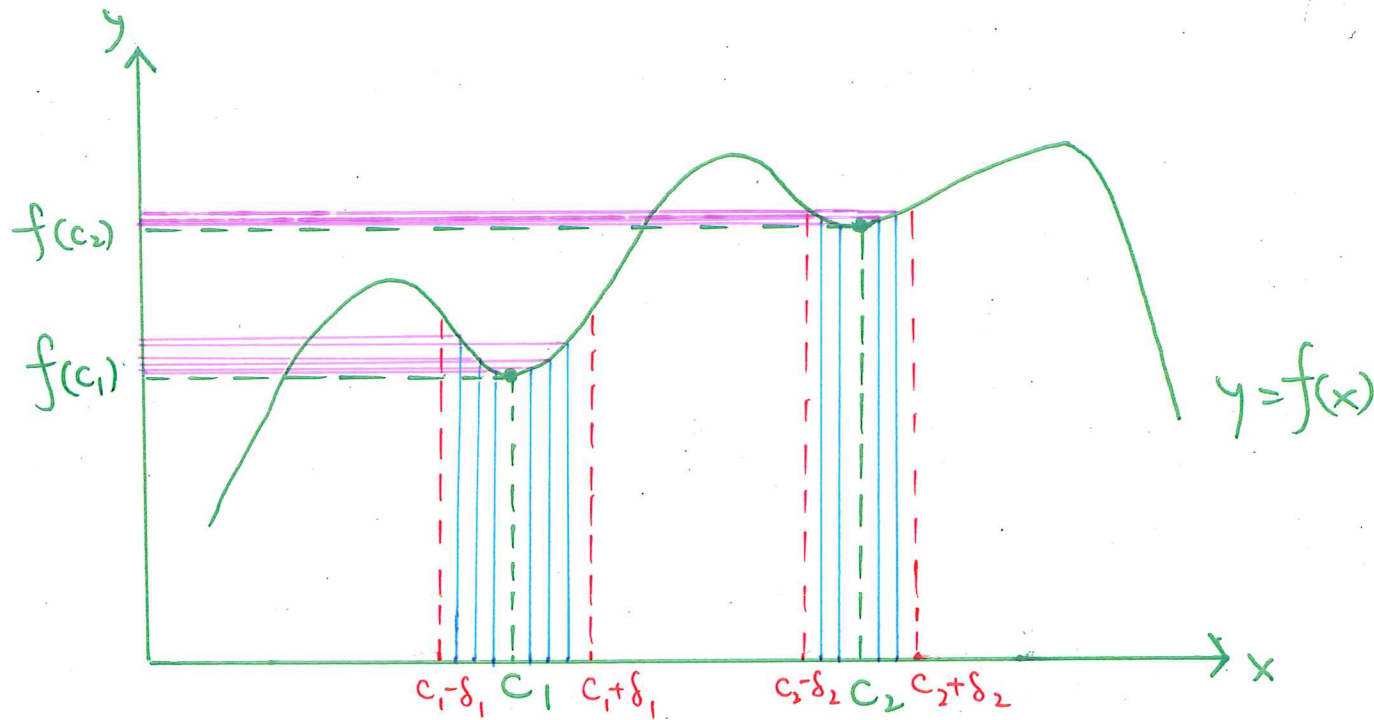
Let f be a function defined on \mathbb{R} , and $c \in \mathbb{R}$.

f attains a relative minimum at $c \iff \exists \delta > 0$ such that for any $x \in \mathbb{R}$, if $|x - c| < \delta$ then $f(x) \geq f(c)$.



Let f be a function defined on \mathbb{R} , and $c \in \mathbb{R}$.

f attains a relative minimum at $c \iff \exists \delta > 0$ such that for any $x \in \mathbb{R}$, if $|x-c| < \delta$ then $f(x) \geq f(c)$.



(b) Let f be a real-valued function on \mathbb{R} , and $c \in \mathbb{R}$.

How to formulate ' f is continuous at c '?

• Formulation in words:

(*) reads: For any $\varepsilon > 0$, there exists some $\delta > 0$ such that for any $x \in \mathbb{R}$, (if $|x-c| < \delta$ then $|f(x)-f(c)| < \varepsilon$).

[Equivalent formulation: For any $\varepsilon > 0$, there exists some $\delta > 0$ such that for any $x \in \mathbb{R}$, ($|x-c| \geq \delta$ or $|f(x)-f(c)| < \varepsilon$).

• Formulation in symbols:

(*) reads: $(\forall \varepsilon > 0) \{ (\exists \delta > 0) \{ (\forall x \in \mathbb{R}) [(|x-c| < \delta) \rightarrow (|f(x)-f(c)| < \varepsilon)] \} \}$

Equivalent formulation: $(\forall \varepsilon > 0) \{ (\exists \delta > 0) \{ (\forall x \in \mathbb{R}) [(|x-c| \geq \delta) \vee (|f(x)-f(c)| < \varepsilon)] \} \}$

How to formulate ' f is not continuous at c '?

• Formulation in symbols: ^{Negation of (*)}

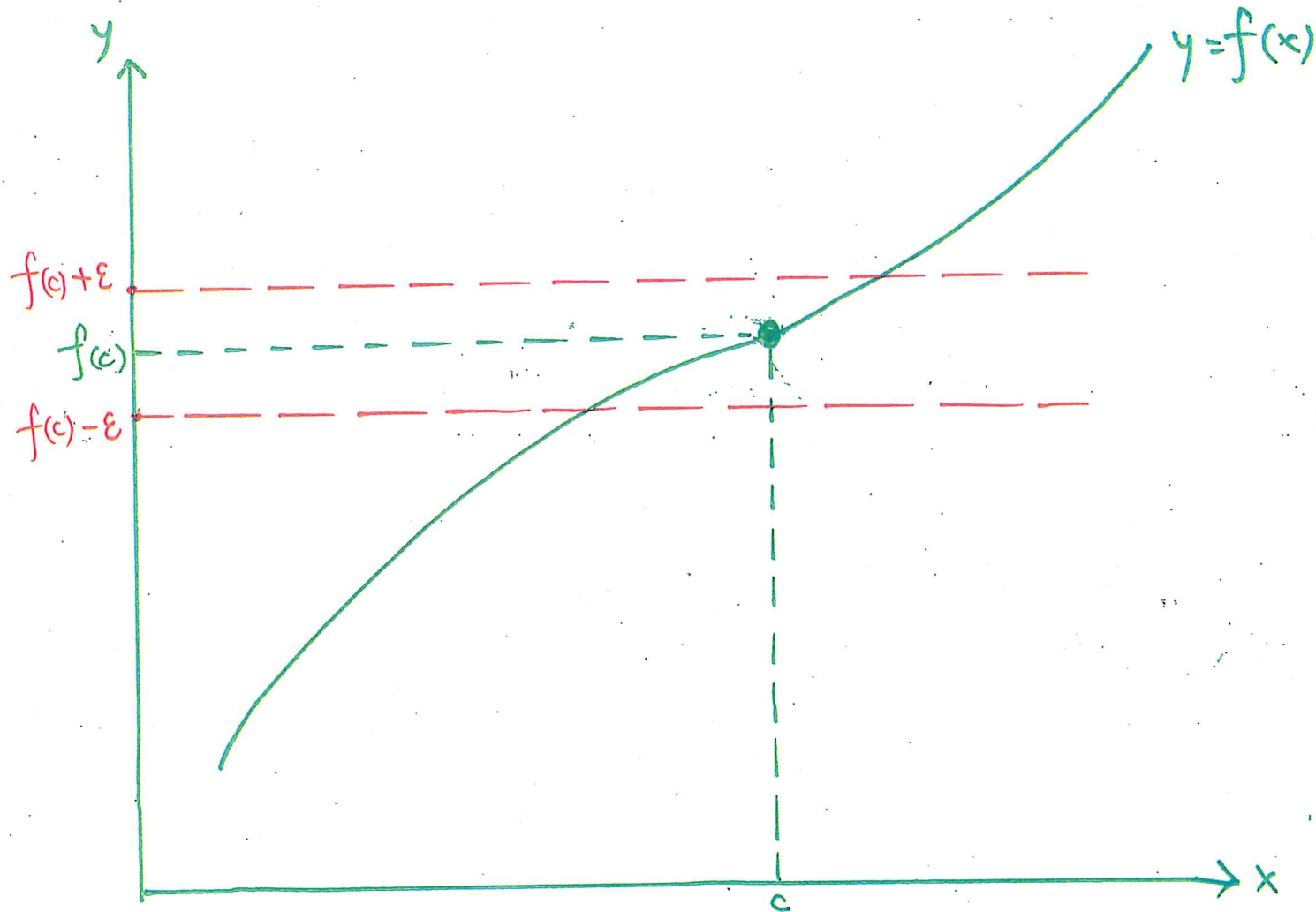
Negation of (*) reads: $(\exists \varepsilon > 0) \{ (\forall \delta > 0) \{ (\exists x \in \mathbb{R}) [(|x-c| < \delta) \wedge (|f(x)-f(c)| \geq \varepsilon)] \} \}$

• Formulation in words:

Negation of (*) reads: There exists some $\varepsilon > 0$ such that for any $\delta > 0$, there exists some $x \in \mathbb{R}$ such that $(|x-c| < \delta$ and $|f(x)-f(c)| \geq \varepsilon$).

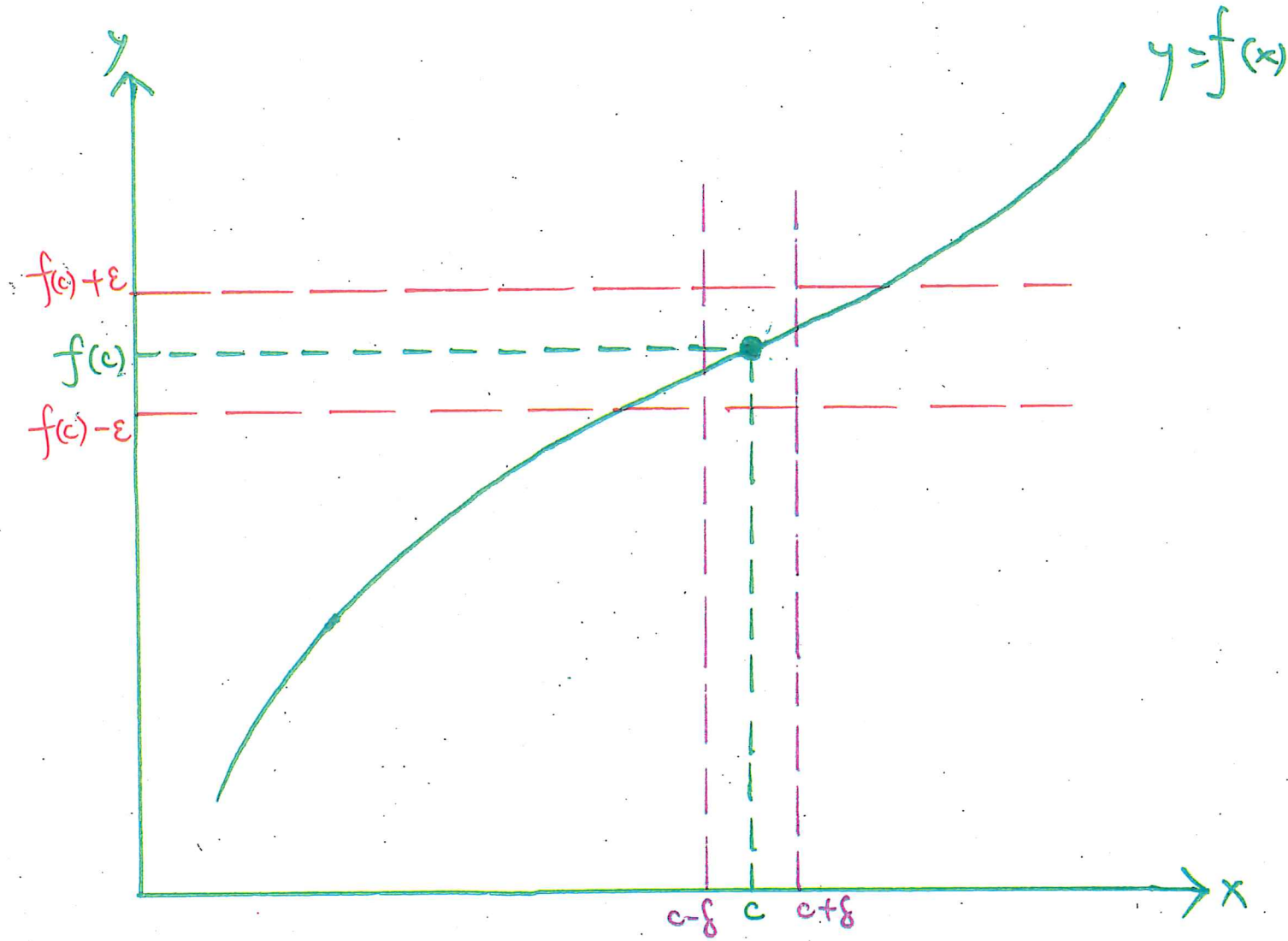
Let f be a function defined on \mathbb{R} , and $c \in \mathbb{R}$.

f is continuous at $c \iff \forall \epsilon > 0, \exists \delta > 0$ such that for any $x \in \mathbb{R}$, if $|x-c| < \delta$ then $|f(x)-f(c)| < \epsilon$.



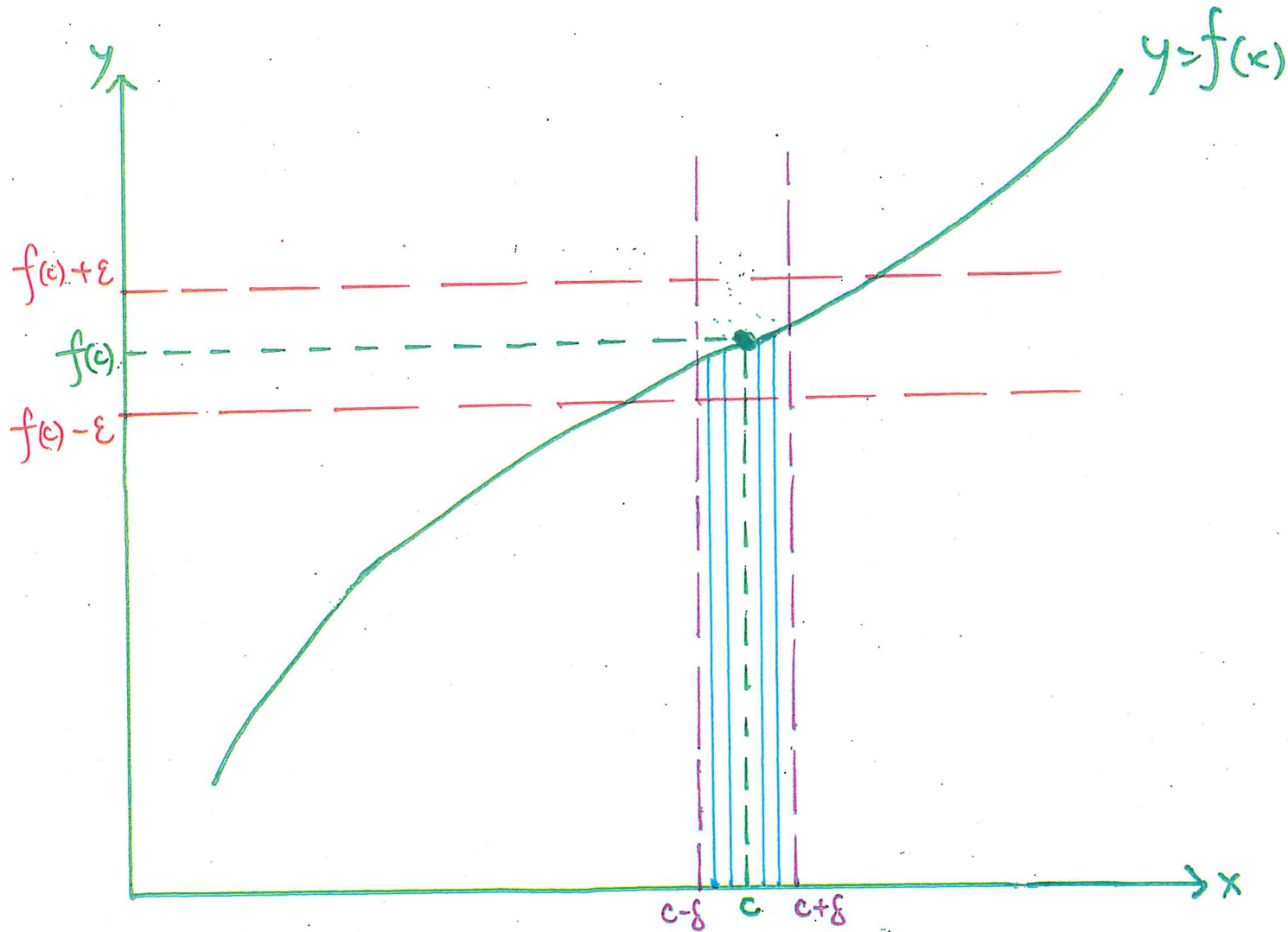
Let f be a function defined on \mathbb{R} , and $c \in \mathbb{R}$.

f is continuous at $c \iff \forall \epsilon > 0, \exists \delta > 0$ such that for any $x \in \mathbb{R}$, if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$.



Let f be a function defined on \mathbb{R} , and $c \in \mathbb{R}$.

f is continuous at c . $\iff \forall \epsilon > 0, \exists \delta > 0$ such that for any $x \in \mathbb{R}$, if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$.



Let f be a function defined on \mathbb{R} , and $c \in \mathbb{R}$.

f is continuous at $c \iff \forall \epsilon > 0, \exists \delta > 0$ such that for any $x \in \mathbb{R}$, if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$.

