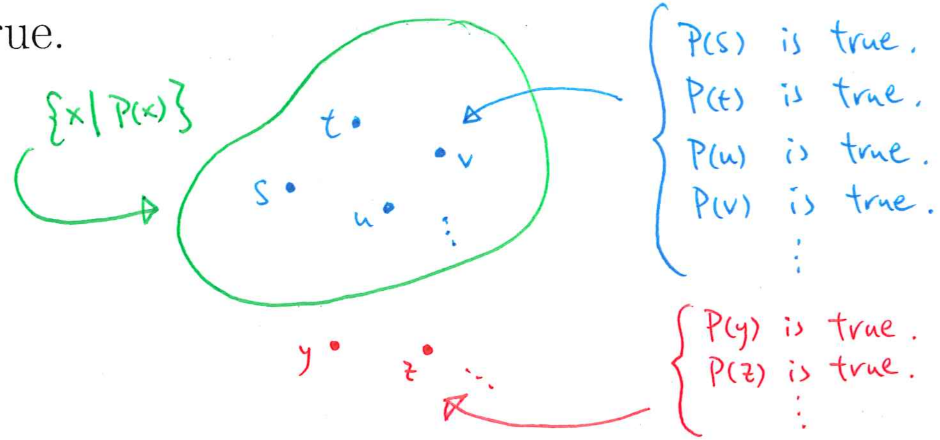
1. Recall **Method of specification (for the construction of sets)**:

Suppose A is a set and $P(x)$ is a predicate with variable x.

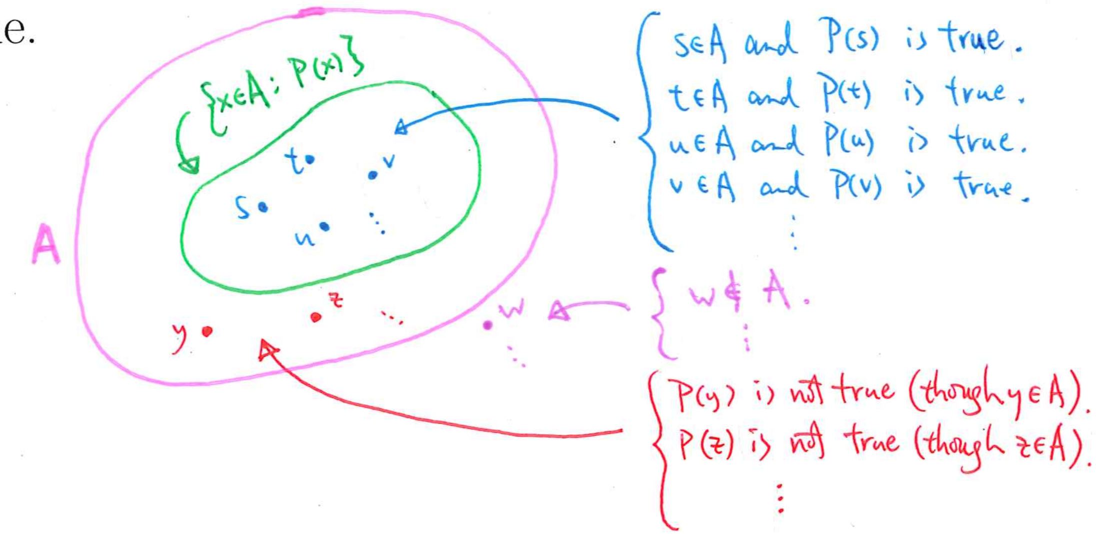
- $\{x \mid P(x)\}$ refers to the set (if it is indeed a set) which contains exactly every object x
  - ∗ for which the statement $P(x)$ is true.



Always remember:
$b \in \{x \mid P(x)\}$
iff
$P(b)$ is true.

- $\{x \in A : P(x)\}$ refers to the set which contains exactly every object x
  - ∗ which is an element of the given set A and
  - ∗ for which the statement $P(x)$ is true.

  By definition it is a subset of A.



Always remember:
$b \in \{x \in A : P(x)\}$
iff
($b \in A$ and $P(b)$ is true.)

2. Recall the notions of **set equality** and ‘**subset relations**’:

- *Let A, B be sets. A is said to be* **equal** *to B if both of the following statements (†), (‡) hold:*

  (†) *For any object x, [if $(x \in A)$ then $(x \in B)$].*

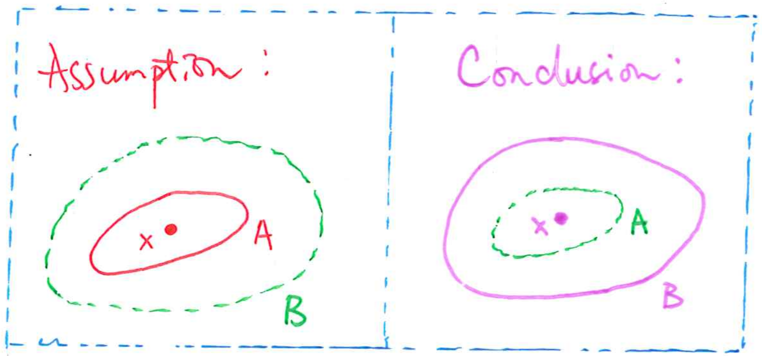
  (‡) *For any object y, [if $(y \in B)$ then $(y \in A)$].*

  We write $A = B$.

- *Let A, B be sets. A is said to be a* **subset** *of B if the following statement (†) holds:*

  (†) *For any object x, [if $(x \in A)$ then $(x \in B)$].* ← For each object x, we have:

  We write $A \subset B$ (or $B \supset A$).

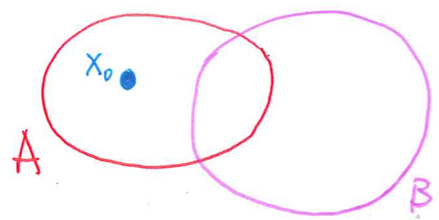


3. **Question.**

What do we mean by 'A is not a subset of B'?

Answer.

A is not a subset of B exactly when some element of A fails to be an element of B.



More useful formulation for the same thing (though formal):

- *There exists some object* x_0 *such that* $(x_0 \in A$ *and* $x_0 \notin B)$.

In this situation, we write $A \not\subset B$.

4. **Example (a).**

Let $C = \{x \mid x = n^4 \text{ for some } n \in \mathbb{N}\}$, $D = \{x \mid x = n^2 \text{ for some } n \in \mathbb{N}\}$.

The following statements hold:

(1) $C \subset D$.

(2) $D \not\subset C$.

Heuristic ideas for the statements:

- C is the set of all biquadratic numbers while D is the set of all square numbers.
- Every biquadratic number is the square of a square number. So we expect '$C \subset D$' to hold.
- There may be some square number which is not a biquadratic number; for instance, the square of a non-square number. So we expect '$D \not\subset C$' to hold.

These are the core ideas in the proofs. They need be present before we write the proofs.

We organize these ideas to give a coherent argument, with reference to the definitions of C, D, '$\subset$'.

5. **Proofs of the statements in Example (a).**

(1) [We want to prove 'for any x, if $x \in C$ then $x \in D$'.]
Pick any object x. Suppose $x \in C$.
[What to deduce? '$x \in D$'. What does it read? 'Unwrap' '$x \in D$' to see what it is.
How to reach '$x \in D$'? 'Unwrap' '$x \in C$' to see what may help us.]

There exists some $m \in \mathbb{N}$ such that $x = m^2$.

Name an appropriate $m \in \mathbb{N}$ for which $x = m^2$

There exists some $n \in \mathbb{N}$ such that $x = n^4$. This has been given to us by assumption.

Since $x \in C$,
there exists some $n \in \mathbb{N}$ such that $x = n^4$.
Take $m = n^2$. Then $x = n^4 = (n^2)^2 = m^2$.
Since $n \in \mathbb{N}$, we have $m \in \mathbb{N}$.
Now we have $x = m^2$ and $m \in \mathbb{N}$.
Hence $x \in D$.
It follows that $C \subset D$. □

[Roughwork:
Any appropriate $m \in \mathbb{N}$ satisfying $x = m^2$?
For such an m, we expect $m^2 = x = n^4$.
So $m = n^2$.]

Proofs of the statements in Example (a).

(2) [Preparation: find out what is to be done.

We want to prove that there exists some x_0 such that $x_0 \in D$ and $x_0 \notin C$. (This is an existence statement.) So we look for an appropriate x_0.

Does our heuristic understanding of C, D in this specific example help us spot a candidate? Is such a candidate a 'good one'?]

[Roughwork: C '=' $\{0, 1, 16, 81, 625, \ldots\}$

D '=' $\{0, 1, 4, 9, 16, 25, \ldots\}$

How about naming $x_0 = 4$?]

Take $x_0 = 4$.

- [Ask: $x_0 \in D$?]

Note that $x_0 = 2^2$ and $2 \in \mathbb{N}$.

Then $x_0 \in D$.

- [Ask: $x_0 \notin C$?]

Claim: $x_0 \notin C$.

We justify this claim (with the help of proof-by-contradiction):

* Suppose it were true that $x_0 \in C$.

Then there would exist some $n \in \mathbb{N}$ such that $x_0 = n^4$.

Now $4 = x_0 = n^4$.

Since $n \in \mathbb{R}$ and $n \geq 0$, we would have $n = \sqrt{2}$.

Now $n = \sqrt{2}$ and $n \in \mathbb{N}$. Contradiction arises.

□

6. Below are other examples similar to Example (a).

Example (b).

Let $C = \{x \mid x = r^4 \text{ for some } r \in \mathbb{Q}\}$, $D = \{x \mid x = r^2 \text{ for some } r \in \mathbb{Q}\}$.

The statements below hold:

(1) $C \subset D$. (2) $D \not\subset C$.

Example (c).

Let $C = \{x \mid x = s + t\sqrt{2} \text{ for some } s, t \in \mathbb{Z}\}$, $D = \{x \mid x = u + v\sqrt{3} \text{ for some } u, v \in \mathbb{Z}\}$.

The statements below hold:

(1) $\mathbb{Z} \subset C \cap D$. (2) $C \not\subset D$. (3) $D \not\subset C$. (4) $C \cap D \subset \mathbb{Z}$. (5) $C \cap D = \mathbb{Z}$.

Example (d).

Let $C = \{x \mid x = s + t\sqrt{2} \text{ for some } s, t \in \mathbb{Q}\}$, $D = \{x \mid x = u + v\sqrt{3} \text{ for some } u, v \in \mathbb{Q}\}$.

The statements below hold:

(1) $\mathbb{Q} \subset C \cap D$. (2) $C \not\subset D$. (3) $D \not\subset C$. (4) $C \cap D \subset \mathbb{Q}$. (5) $C \cap D = \mathbb{Q}$.

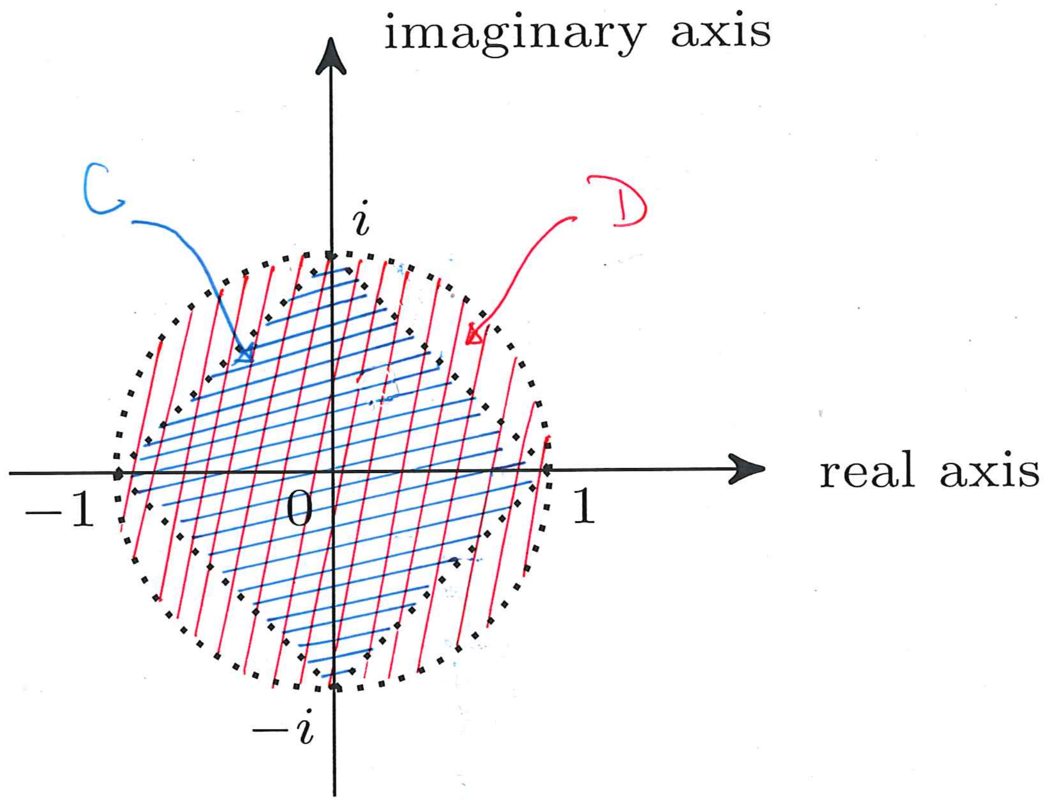
7. **Example (e).**

Let $C = \{\zeta \in \mathbb{C} : |\mathsf{Re}(\zeta)| + |\mathsf{Im}(\zeta)| < 1\}$, $D = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$.

The statements below hold:

(1) $C \subset D$. (2) $D \not\subset C$.

Heuristic ideas for the statements, which can be visualized using the Argand plane:



8. **Proofs of the statements in Example (e).**

(1) [We want to prove 'for any $\zeta \in \mathbb{C}$, if $\zeta \in C$ then $\zeta \in D$'.]

Pick any object ζ. Suppose $\zeta \in C$.

[What to deduce? '$\zeta \in D$'. What does it read? '$|\zeta| < 1$.' How to reach '$\zeta \in D$'?

Find out what '$\zeta \in C$' reads: it is $|\mathsf{Re}(\zeta)| + |\mathsf{Im}(\zeta)| < 1$.]

Then $|\mathrm{Re}(\zeta)| + |\mathrm{Im}(\zeta)| < 1$. —— (☆)

[How does $|\zeta|$ link up with $|\mathrm{Re}(\zeta)|$, $|\mathrm{Im}(\zeta)|$?]

We have

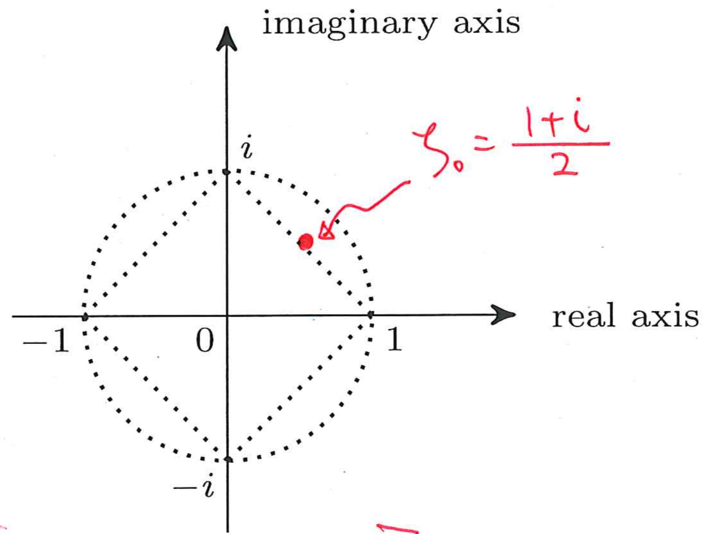
$$|\zeta|^2 = (\mathrm{Re}(\zeta))^2 + (\mathrm{Im}(\zeta))^2$$
$$= |\mathrm{Re}(\zeta)|^2 + |\mathrm{Im}(\zeta)|^2$$
$$= |\mathrm{Re}(\zeta)| \cdot |\mathrm{Re}(\zeta)| + |\mathrm{Im}(\zeta)| \cdot |\mathrm{Im}(\zeta)|$$
$$\leq |\mathrm{Re}(\zeta)| \cdot 1 + |\mathrm{Im}(\zeta)| \cdot 1$$
$$= |\mathrm{Re}(\zeta)| + |\mathrm{Im}(\zeta)| < 1 \quad \text{by (☆)}.$$

Then $|\zeta| < 1$. Therefore $\zeta \in D$.

It follows that $C \subset D$.

Proofs of the statements in Example (e).

(2) [Preparation: find out what is to be done. We want to prove that there exists some ζ_0 such that $\zeta_0 \in D$ and $\zeta_0 \notin C$. (This is an existence statement.) So we look for an appropriate ζ_0. $\cdots$]



Ask:

- $\zeta_0 \in D$?
- $\zeta_0 \notin C$?

- Take $\zeta_0 = \dfrac{1+i}{2}$.
- Note that

$$|\zeta_0|^2 = (\mathrm{Re}(\zeta_0))^2 + (\mathrm{Im}(\zeta_0))^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1.$$

Then $|\zeta_0| < 1$.

Therefore $\zeta_0 \in D$.

- Note that

$$|\mathrm{Re}(\zeta_0)| + |\mathrm{Im}(\zeta_0)| = \frac{1}{2} + \frac{1}{2} = 1.$$

Then $\zeta_0 \notin C$.

It follows that $D \nsubseteq C$.

9. Below are other examples similar to Example (e).

Example (f).

Let $C = \{\zeta \in \mathbb{C} : |\zeta - 1| \leq 1\}, D = \{\zeta \in \mathbb{C} : |\zeta| \leq 2\}$.

The statements below hold:

(1) $C \subset D$. (2) $D \not\subset C$.

Example (g).

Let $C = \{\zeta \in \mathbb{C} : \mathsf{Re}(\zeta) \geq 0\}, D = \{\zeta \in \mathbb{C} : \mathsf{Im}(\zeta) \geq 0\}, E = \{\zeta \in \mathbb{C} : |\zeta - 1 - i| \leq 1\}$.

The statements below hold:

(1) $E \subset C \cap D$. (2) $C \not\subset D$. (3) $D \not\subset C$.

Example (h).

Let $C = \{\zeta \in \mathbb{C} : |\zeta - 4| < 5\}, D = \{\zeta \in \mathbb{C} : |\zeta + 4| < 5\}, E = \{\zeta \in \mathbb{C} : |\zeta| < 3\}$.

The statements below hold:

(1) $C \not\subset D$. (2) $D \not\subset C$. (3) $E \not\subset C$. (4) $E \not\subset D$. (5) $E \subset C \cup D$.

10. **Example (i).**

Let G be an $(m \times n)$-matrix with real entries, and H be an $(n \times p)$-matrix with real entries. The statements below hold:

(1) *The null space of H is a subset of the null space of GH.*

(2) *Suppose the null space of G is $\{\mathbf{0}_n\}$. Then the null space of GH is a subset of the null space of H.*

Remark. *The null space $\mathcal{N}(K)$ of a $(p \times q)$-matrix K with real entries is defined by $\mathcal{N}(K) = \{\mathbf{x} \in \mathbb{R}^q : K\mathbf{v} = \mathbf{0}_p\}$.*

11. **Proofs of the statements in Example (i).**

(1) [We want to prove 'for any $\mathbf{x} \in \mathbb{R}^p$, if $\mathbf{x} \in \mathcal{N}(H)$ then $\mathbf{x} \in \mathcal{N}(GH)$'.]

Pick any $\mathbf{x} \in \mathbb{R}^p$. Suppose $\mathbf{x} \in \mathcal{N}(H)$.

[What to deduce? '$\mathbf{x} \in \mathcal{N}(GH)$'. What does it read? '$(GH)\mathbf{x} = \mathbf{0}_m$.' How to reach '$(GH)\mathbf{x} = \mathbf{0}_m$'? Find out what '$\mathbf{x} \in \mathcal{N}(H)$' reads: it is '$H\mathbf{x} = \mathbf{0}_n$'.]

Then by the definition of $\mathcal{N}(H)$, we have $H\mathbf{x} = \mathbf{0}_n$.

Therefore $(GH)\mathbf{x} = G(H\mathbf{x}) = G\mathbf{0}_n = \mathbf{0}_m$.

Hence, by the definition of $\mathcal{N}(GH)$, we have $\mathbf{x} \in \mathcal{N}(GH)$.

It follows that $\mathcal{N}(H) \subset \mathcal{N}(GH)$.

(2) Suppose the null space of G is $\{\mathbf{0}_n\}$.

[We want to deduce, under the above assumption, that '$\mathcal{N}(GH) \subset \mathcal{N}(H)$', which reads: 'for any $\mathbf{x} \in \mathbb{R}^p$, if $\mathbf{x} \in \mathcal{N}(GH)$ then $\mathbf{x} \in \mathcal{N}(H)$'.]

Pick any $\mathbf{u} \in \mathbb{R}^p$. Suppose $\mathbf{u} \in \mathcal{N}(GH)$.

[What to deduce? '$\mathbf{u} \in \mathcal{N}(H)$'. What does it read? '$H\mathbf{u} = \mathbf{0}_n$.' How to reach '$H\mathbf{u} = \mathbf{0}_n$'? Find out what '$\mathbf{u} \in \mathcal{N}(GH)$' reads: it is '$(GH)\mathbf{u} = \mathbf{0}_m$'.]

Then by the definition of $\mathcal{N}(GH)$, we have $G(H\mathbf{u}) = (GH)\mathbf{u} = \mathbf{0}_m$.

Therefore, by the definition of $\mathcal{N}(G)$, we have $H\mathbf{u} \in \mathcal{N}(G)$.

Since $\mathcal{N}(G) = \{\mathbf{0}_m\}$, we have $H\mathbf{u} \in \{\mathbf{0}_m\}$. Then $H\mathbf{u} = \mathbf{0}_n$.

Therefore, by the definition of $\mathcal{N}(H)$, we have $\mathbf{u} \in \mathcal{N}(H)$.

It follows that $\mathcal{N}(GH) \subset \mathcal{N}(H)$.

12. **Example (j).**

Let S, T be subsets of $\mathbb{R}^n$, G be an $(m \times n)$-matrix with real entries, and H be an $(n \times p)$-matrix with real entries.

Define
$S' = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = G\mathbf{x} \text{ for some } \mathbf{x} \in S\}$,
$T' = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = G\mathbf{x} \text{ for some } \mathbf{x} \in T\}$.

Define
$S^* = \{\mathbf{u} \in \mathbb{R}^p : \mathbf{x} = H\mathbf{u} \text{ for some } \mathbf{x} \in S\}$,
$T^* = \{\mathbf{u} \in \mathbb{R}^p : \mathbf{x} = H\mathbf{u} \text{ for some } \mathbf{x} \in T\}$.

The statements below hold:

(1) *Suppose S is a subset of T. Then S' is a subset of T'.*

(2) *Suppose S is a subset of T. Then S^* is a subset of T^*.*

13. **Proofs of the statements in Example (j)**.

(1) Suppose S is a subset of T.

[We want to deduce, under the above assumption, that 'S' is a subset of T' ', which reads: 'for any $\mathbf{y} \in \mathbb{R}^m$, if $\mathbf{y} \in S'$ then $\mathbf{y} \in T'$ '.]

[Recall what S' and T' are:
$S' = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = G\mathbf{x} \text{ for some } \mathbf{x} \in S\}$,
$T' = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = G\mathbf{x} \text{ for some } \mathbf{x} \in T\}$, in which G is some fixed $(m \times n)$-matrix.]

Pick any object $\mathbf{y} \in \mathbb{R}^m$. Suppose $\mathbf{y} \in S'$.

[What to deduce? '$\mathbf{y} \in T'$ '. What does it read? 'Unwrap' '$\mathbf{y} \in T'$ ' to see what it is. How to reach '$\mathbf{y} \in T'$ '? 'Unwrap' '$\mathbf{y} \in S'$ ' to see what may help us.]

Then by the definition of S', there exists some $\mathbf{x} \in S$ such that $\mathbf{y} = G\mathbf{x}$.

Note that $\mathbf{x} \in S$, and by assumption S is a subset of T. Then, by the definition of subset relations, $\mathbf{x} \in T$.

Therefore $\mathbf{x} \in T$ and $\mathbf{y} = G\mathbf{x}$ for the same $\mathbf{x}, \mathbf{y}$.
Hence, by the definition of T', we have $\mathbf{y} \in T'$.

It follows that $S' \subset T'$.

(2) Suppose S is a subset of T.

[We want to deduce, under the above assumption, that that 'S^* is a subset of T^* ', which reads:'for any $\mathbf{u} \in \mathbb{R}^p$, if $\mathbf{u} \in S^*$ then $\mathbf{u} \in T^*$ '.]

[Recall what S^* and T^* are:
$S^* = \{\mathbf{u} \in \mathbb{R}^p : \mathbf{x} = H\mathbf{u} \text{ for some } \mathbf{x} \in S\}$,
$T^* = \{\mathbf{u} \in \mathbb{R}^p : \mathbf{x} = H\mathbf{u} \text{ for some } \mathbf{x} \in T\}$, in which H is some fixed $(m \times p)$-matrix.]

Pick any object $\mathbf{u} \in \mathbb{R}^p$. Suppose $\mathbf{u} \in S^*$.

[What to deduce? '$\mathbf{u} \in T^*$ '. What does it read? 'Unwrap' '$\mathbf{u} \in T^*$ ' to see what it is. How to reach '$\mathbf{u} \in T^*$ '? 'Unwrap' '$\mathbf{u} \in S^*$ ' to see what may help us.]

Then by the definition of S^*, there exists some $\mathbf{x} \in S$ such that $\mathbf{x} = H\mathbf{u}$.

Note that $\mathbf{x} \in S$, and by assumption S is a subset of T. Then, by the definition of subset relations, $\mathbf{x} \in T$.

Therefore $\mathbf{x} \in T$ and $\mathbf{x} = H\mathbf{u}$, for the same $\mathbf{x}, \mathbf{u}$.
Hence, by the definition of T^*, we have $\mathbf{u} \in T^*$.

It follows that $S^* \subset T^*$.