MATH1050 Applications of logic in mathematics

1. **Modus Ponens and the symbols '$\Longrightarrow$', '$\Longleftrightarrow$'.**

   Inspect the truth table for two given statements P, Q together with the compound statements $P \to Q$, $(P \to Q) \wedge P$, $[(P \to Q) \wedge P] \to Q$:

P	Q	$P \to Q$	$(P \to Q) \wedge P$	$[(P \to Q) \wedge P] \to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

   As demonstrated in the truth table above, $[(P \to Q) \wedge P] \to Q$ is indeed a tautology. This tautology is known as modus ponens.

   Modus ponens is so widely used in mathematics (and elsewhere) that you might not be conscious of it. Some mathematicians choose to indicate its application in a piece of mathematics with the help of the symbol '$\Longrightarrow$', or a chain of this symbol '$\Longrightarrow$'. The way it should be understood is described below:

   - Whenever '$G \Longrightarrow H$' is written (in which G, H are two concretely given mathematical statements), the reader is informed that, the statement G is (known/asserted to be) true and the conditional '$G \to H$' is also (known to be) true, and so by Modus Ponens it may be concluded that the statement H is true.
   - Whenever '$G \Longrightarrow H \Longrightarrow J$' is written (in which G, H, J are three concretely given mathematical statements), the reader is informed that, the statement G is (known/asserted to be) true and the conditionals '$G \to H$', '$H \to J$' are also (known to be) true, and so by two successive applications of Modus Ponens it may be concluded that the statement J is true.
     So forth and so on.

   So whenever there is the temptation to write 'blah-blah-blah $\Longrightarrow$ bleh-bleh-bleh' or

   $$\begin{array}{cc} & \text{blah-blah-blah} \\ \Longrightarrow & \text{bleh-bleh-bleh} \\ \Longrightarrow & \text{blih-blih-blih} \\ \vdots & \vdots \\ \Longrightarrow & \text{bloh-bloh-bloh,} \end{array}$$

   think whether you mean the above.

   Given two mathematical statements G, H, when '$G \Longleftrightarrow H$' is written, the reader is informed that '$G \Longrightarrow H$', '$H \Longrightarrow G$' are separately and independently asserted.

2. **What is a 'direct proof'?**

   Consider a mathematical statement of the form '$(P_1 \wedge P_2 \cdots \wedge P_m) \longrightarrow (Q_1 \wedge Q_2 \wedge \cdots \wedge Q_n)$'. A 'direct proof' for such a conditional is a list of mathematical statements organized with respect to logic, so that the assumptions $P_1, P_2, \cdots, P_m$ appear in the beginning and the conclusions $Q_1, , Q_2, \cdots, Q_n$ appear at the end. Each statement in the list is a 'logical consequence', according to some logical equivalence or a rule of inference, of one or several statements preceding the statement concerned. Those preceding statements provide the 'mathematical reason(s)' for the statement concerned.

   Most often we do not write proofs in this way. We write proofs that, in principle, can be organized such a list. This is because we cannot read like machines in practice.

   Example. (Refer to the Handout *Examples of simple inequalities justified using 'direct proofs'*.) Consider the statement below:

   - '*Let x, y be positive real numbers. Suppose $x^2 > y^2$. Then $x > y$.*'

   Its proof is given by:

   - Let x, y be positive real numbers. Suppose $x^2 > y^2$. Then $x^2 - y^2 > 0$.
     Note that $x^2 - y^2 = (x - y)(x + y)$. Then $(x - y)(x + y) > 0$.
     Therefore ($x - y > 0$ and $x + y > 0$) or ($x - y < 0$ and $x + y < 0$).
     Since $x > 0$ and $y > 0$, we have $x + y > 0$. Then $x - y > 0$ and $x + y > 0$. In particular $x - y > 0$. Therefore $x > y$.

The statement to be proved is of the form $(P_1 \wedge P_2) \longrightarrow Q$, in which

- P_1 stands for 'x, y are positive real numbers',
- P_2 stands for '$x^2 > y^2$', and
- Q stands for $x > y$.

The statements in the proof can be organized a 'very formal proof', which is the list below:

I. Let x, y be positive real numbers. [Assumption.]

II. Suppose $x^2 > y^2$. [Assumption.]

III. $x^2 - y^2 > 0$. [**II**.]

IV. $x^2 - y^2 = (x - y)(x + y)$. [Properties of the reals.]

V. $(x - y)(x + y) > 0$. [**III**, **IV**.]

VI $(x - y > 0$ and $x + y > 0)$ or $(x - y < 0$ and $x + y < 0)$. [**V**, properties of the reals.]

VII. $x + y > 0$ [**I**.]

VIII. $x - y > 0$. [**VI**, **VII**.]

IX. $x > y$. [**VIII**.]

The assumptions P_1, P_2 appear at the top of the list: they are Statements (**I**), (**II**).

The conclusion Q appears at the bottom: it is Statement (**IX**).

We give the detail on how logical equivalence and the rules of inference work in the rest of the list:

- Statement (**III**) is a consequence of Statement (**II**) in this sense:

  According to the properties of the reals, if the statement '$x^2 > y^2$' holds then the statement '$x^2 - y^2 > 0$' holds.
  The statement '$x^2 > y^2$' holds by assumption.
  Therefore, by **Modus Ponens**, the statement '$x^2 - y^2 > 0$' holds.

- Statement (**IV**) is a 'known' property of the reals.
- Statement (**V**) is a consequence of Statements (**III**), (**IV**) in this sense:

  According to the properties of the reals, if the statement '$x^2 - y^2 > 0$' holds and the statement '$x^2 - y^2 = (x - y)(x + y)$' holds then the statement '$(x - y)(x + y) > 0$' holds.
  The statement '$x^2 - y^2 > 0$' holds.
  The statement '$x^2 - y^2 = (x - y)(x + y)$' holds.
  Therefore, by **adjunction**, the statement '$x^2 - y^2 > 0$' holds and the statement '$x^2 - y^2 = (x - y)(x + y)$' holds.
  Hence, by **Modus Ponens**, the statement '$(x - y)(x + y) > 0$' holds.

- Statement (**VI**) is a consequence of Statement (**V**) in this sense:

  According to the properites of the reals, if the statement '$(x - y)(x + y) > 0$' holds then the statement '$(x - y > 0$ and $x + y > 0)$ or $(x - y < 0$ and $x + y < 0)$' holds.
  The statement '$(x - y)(x + y) > 0$' holds.
  Therefore, by **Modus Ponens**, the statement '$(x - y > 0$ and $x + y > 0)$ or $(x - y < 0$ and $x + y < 0)$' holds.

- Statement (**VII**) is a conseqence of Statement (**I**) in this sense:

  According to the properites of the reals, if the statement '$x > 0$ and $y > 0$' holds then the statement '$x + y > 0$' holds.
  The statement '$x > 0$ and $y > 0$' holds.
  Therefore, by **Modus Ponens**, the statement '$x + y > 0$' holds.

- Statement (**VIII**) is a consequence of Statements (**VI**), (**VII**) in this sense:

  According to the properties of the reals, if the statement '$x + y > 0$' holds then the statement '$x + y \geq 0$' holds.
  According to the properties of the reals, if the statement '$x + y \geq 0$' holds then the negation of the statement '$x + y < 0$' holds.
  Hence, by **hypothetical syllogisms**, if the statement '$x + y > 0$' holds then the negation of the statement '$x + y < 0$' holds.

Recall that $x+y>0$ indeed holds. Then by **Modus Ponens**, the negation of the statement '$x+y<0$' holds.
Now, by **addition**, the disjunction of the negation of the statement '$x+y<0$' and the negation of the statement '$x-y<0$' holds.
Then, by **De Morgan's Law**, the negation of the conjunction of the statements '$x+y<0$', '$x-y<0$' holds.
Recall that the statement '($x-y>0$ and $x+y>0$) or ($x-y<0$ and $x+y<0$)' holds.
Therefore, by **Modus Tollendo Ponens**, the statement '$x-y>0$ and $x+y>0$' holds.
Hence, by **Simplification**, the statement '$x-y>0$' holds.

- Statement (**IX**) is a consequence of Statement (**VIII**) in this sense:

  According to the properties of the reals, if the statement '$x-y>0$' holds then the statement '$x>y$' holds.
  The statement '$x-y>0$' holds.
  Therefore, by **Modus Ponens**, the statement '$x>y$' holds.

In light of the above, we should have expanded the 'very formal proof' into a 'very very formal proof' (which incorporates each 'sub-step' that we have mentioned):

I. Let x, y be positive real numbers. [Assumption.]

II. Suppose $x^2>y^2$. [Assumption.]

S1. If $x^2>y^2$ then $x^2-y^2>0$. [Properties of the reals.]

III. $x^2-y^2>0$. [**II**, **S1**, **Modus Ponens**.]

IV. $x^2-y^2=(x-y)(x+y)$. [Properties of the reals.]

S2. $x^2-y^2>0$ and $x^2-y^2=(x-y)(x+y)$. [**III**, **IV**, **Adjunction**.]

S3. If ($x^2-y^2>0$ and $x^2-y^2=(x-y)(x+y)$) then $(x-y)(x+y)>0$. [Properties of the reals.]

V. $(x-y)(x+y)>0$. [**S2**, **S3**, **Modus Ponens**.]

S4. If $(x-y)(x+y)>0$ then [($x-y>0$ and $x+y>0$) or ($x-y<0$ and $x+y<0$)]. [Properties of the reals.]

VI ($x-y>0$ and $x+y>0$) or ($x-y<0$ and $x+y<0$). [**V**, **S4**, **Modus Ponens**.]

S5. If ($x>0$ and $y>0$) then $x+y>0$ [Properties of the reals.]

VII. $x+y>0$ [**I**, **S5**, **Modus Ponens**.]

S6. If $x+y>0$ then $x+y\geq 0$. [Properties of the reals.]

S7. If $x+y\geq 0$ then (it is not true that $x+y<0$). [Properties of the reals.]

S8. If $x+y>0$ then (it is not true that $x+y<0$). [**S6**, **S7**, **Hypothetical Syllogism**.]

S9. It is not true that $x+y<0$. [**VII**, **S8**, **Modus Ponens**.]

S10. (It is not true that $x+y<0$) or (it is not true that $x-y<0$). [**S9**, **Addition**.]

S11. It is not true that ($x+y<0$ and $x-y<0$). [**S10**, **De Morgan's Laws**.]

S12. $x-y>0$ and $x+y>0$. [**VI**, **S10**, **Modus Tollendo Ponens**.]

VIII. $x-y>0$. [**S12**, **Simplification**.]

S13. If $x-y>0$ then $x>y$. [Properties of the reals.]

IX. $x>y$. [**VIII**, **S13**, **Modus Ponens**.]

If you do not realized why we do not want to write proofs in such a 'very formal' style, you should have done so by now. This, however, did not prevent Russell and Whitehead from writing *Principia Mathematica* (in three volumes, each of several hundred pages), in such a style, which they hoped to establish all the mathematics known by their time (about 100 years ago) from what they regard to be most basic.

3. **What is 'solving a system of equations/inequalities with unknown(s) in the set so-and-so'?**

Whenever you are 'solving a system of m equations/inequalities with unknonwn(s) in the set so-and-so', you are in fact attempting to prove a statement, about objects in the set so-and-so, of the form $(P_1 \wedge P_2 \wedge \cdots \wedge P_m) \longleftrightarrow Q$, in which:

- $P_1, P_2, \cdots, P_m$ are the statements obtained after substituting the objects under consideration into the m equations/inequalities, and
- Q is a statement which gives the explicit description of the objects and is called the solution of the system of equations/inequalities.

In principle, the argument should be made up of two parts (or can be read as such):

- First prove the statement

  (†) $(P_1 \wedge P_2 \wedge \cdots \wedge P_m) \longrightarrow Q$.

  This is a list with the equations/inequalities $P_1, P_2, \cdots, P_m$ at the top and with the solution Q at the bottom. In very simple situation, this is a 'chain' of calculations which 'demonstrates' how you obtain the solution.
- Next prove the statement

  (‡) $Q \longrightarrow (P_1 \wedge P_2 \wedge \cdots \wedge P_m)$.

  You start with the solution Q and deduce the statements $P_1, P_2, \cdots, P_m$: this is the 'checking the solution'.

Examples.

(a) When we solve the equation $x^2 - 3x + 2 = 0$ with unknown x in $\mathbb{R}$, we present this 'chain' of calculation:

$$\begin{array}{rcl} x^2 - 3x + 2 & = & 0 \\ (x-1)(x-2) & = & 0 \\ x - 1 = 0 & \text{or} & x - 2 = 0 \\ x = 1 & \text{or} & x = 2 \end{array}$$

This 'chain' of calculation is in fact part of a proof for the statement (⋆):

(⋆) Let x be a real number. $x^2 - 3x + 2 = 0$ iff ($x = 1$ or $x = 2$).

The proof of (⋆) is given below:

Let x be a real number.

- Suppose $x^2 - 3x + 2 = 0$.
  Then $(x-1)(x-2) = 0$.
  Therefore $x - 1 = 0$ or $x - 2 = 0$.
  Hence $x = 1$ or $x = 2$.
  It follows that if $x^2 - 3x + 2 = 0$ then ($x = 1$ or $x = 2$).
- Suppose ($x = 1$ or $x = 2$).
  Then $x - 1 = 0$ or $x - 2 = 0$.
  Therefore $x^2 - 3x + 2 = (x-1)(x-2) = 0$.
  It follows that if ($x = 1$ or $x = 2$) then $x^2 - 3x + 2 = 0$.

What we have done is verified that for each fixed real number x, the statement $P \longleftrightarrow (Q_1 \vee Q_2)$ is true, in which P stands for '$x^2 - 3x + 2 = 0$', and Q_1, Q_2 stand for '$x = 1$', '$x = 2$' respectively.

Note that the argument for $P \longrightarrow (Q_1 \vee Q_2)$ gives exactly the 'chain' of calculations which we 'demonstrate' how to obtain the solution $x = 1$ or $x = 2$ from the equation $x^2 - 3x + 2 - 0$. In the context of the proof, what we have done in the first part is to obtain 'candidate solutions' for the equation. There is, however, no guarantee that a 'candidate solution' is indeed a solution. Hence we argue for $(Q_1 \vee Q_2) \longrightarrow P$. This is a more formal way of presenting the 'checking the solution' that we might be reminded to write as beginners:

Put $x = 1$ into $x^2 - 3x + 2 = 0$. LHS = RHS.
Put $x = 2$ into $x^2 - 3x + 2 = 0$. LHS = RHS.

This is to verify that both 'candidate solutions' are indeed solutions of the equation $x^2 - 3x + 2 = 0$.

(b) When we solve the equation $3x = |2x - 3|$ with unknown x in $\mathbb{R}$, we present this 'chain' of calculation:

$$\begin{array}{rcl} 3x & = & |2x - 3| \\ 3x = 2x - 3 & \text{or} & 3x = -(2x - 3) \\ x = -3 & \text{or} & x = \dfrac{3}{5} \end{array}$$

Then we proceed with 'checking the (candidate) solution':

- $3(-3) = -9 \neq 9 = |2 \cdot (-3) - 3|$.
- $3 \cdot \dfrac{3}{5} = \dfrac{9}{5} = |2 \cdot \dfrac{3}{5} - 3|$.

We then conclude: the only solution of the equation $3x = |2x-3|$ with unknown x in $\mathbb{R}$ is '$x = \frac{3}{5}$'

The 'chain' of calculations correspond to the argument for the statement $P \longrightarrow (Q_1 \vee Q_2)$, where P stands for '$3x = |2x-3|$', Q_1 stands for '$x = -3$' and Q_2 stands for '$x = \frac{3}{5}$'.

This time '$Q_1 \longrightarrow P$' is true while '$Q_2 \longrightarrow P$' is false.

The 'checking the solutions' correspond to the argument for the statement $Q_1 \longrightarrow P$ and the negation of the statement $Q_2 \longrightarrow P$.

Combined together, we are arguing for the statement $P \longleftrightarrow Q_1$, in which Q_1 stands for the solution of the equation $3x = |2x-3|$ with unknown x in $\mathbb{R}$.

4. **What is the mechanism behind the proof-by-contradiction method?**

Recall that $P \to Q$ is the same as $\sim[P \wedge (\sim Q)]$, which is the same as 'the statement $P \wedge (\sim Q)$ is false'. To justify one of them is the same as to justify the other. This is the logical foundation of 'proof-by-contradiction'.

Example. (Refer to the Handout *Examples of proofs-by-contradiction.*)

Consider the statement below:

- 'Let $a, b \in \mathbb{Q}$. Suppose $b \neq 0$. Then $a + b\sqrt{2}$ is irrational.'

A proof-by-contradiction argument for this statement is given by:

- Suppose a, b are rational numbers and $b \neq 0$.
  Suppose it were true that $a + b\sqrt{2}$ was a rational number. Write $r = a + b\sqrt{2}$.
  Since a, r were rational numbers and $b\sqrt{2} = r - a$, $b\sqrt{2}$ would a rational number.
  Since b is a non-zero rational number and $\sqrt{2} = \frac{b\sqrt{2}}{b}$, $\sqrt{2}$ would be a rational number.
  But $\sqrt{2}$ is an irrational number.
  This is a contradiction.
  Hence our assumption that $a + b\sqrt{2}$ was a rational number is false.
  $a + b\sqrt{2}$ is an irrational number.

We recognize that this statement is of the form $(U \wedge V \wedge W) \to S$, where

- U stands for '$a \in \mathbb{Q}$',
- V stands for '$b \in \mathbb{Q}$',
- W stands for '$b \neq 0$' and
- S stands for '$a + b\sqrt{2}$ is irrational'.

To apply proof-by-contradiction to argue for the statement '$(U \wedge V \wedge W) \to S$ is true', we argue for the statement '$\sim[(U \wedge V \wedge W) \to S]$ is false', which is the same as '$U \wedge V \wedge W \wedge (\sim S)$ is false'. Hence we start with:

'Let $a, b \in \mathbb{Q}$. Suppose $b \neq 0$. Suppose it were true that $a + b\sqrt{2}$ was rational.'

Then we try to show the falsity of '$U \wedge V \wedge W \wedge (\sim S)$' by deducing something 'wrong' (which we call a contradiction), namely a statement which is known to be false. In this example, this false statement is chosen to be the statement '$\sqrt{2}$ is rational', which we denote by C from now on. (This part of the argument from 'suppose a, b are rational numbers ... ' to '$\sqrt{2}$ would be a rational number' can be put into a list of statement in which U, V, W appear at the top and C appears at the bottom, and in which every statement in the middle logically follows from statements which appear above it.)

At the end of the process, we conclude that if $U \wedge V \wedge W$ is true then S is true.

What is the mechanism behind the whole process? Consider the truth table below, with only the row crucial to our consideration displayed:

U	V	W	S	$U \wedge V \wedge W$	$\sim S$	$U \wedge V \wedge W \wedge (\sim S)$	C	$[U \wedge V \wedge W \wedge (\sim S)] \to C$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
T	T	T	???	T	??	?	F	T

U, V, W are assumed to be true. The body of the argument itself amounts to telling us that the statement '$U \wedge V \wedge W \wedge (\sim S) \to C$' is true. Since C is (known to be) false, the only possibility is that the statement '$U \wedge V \wedge W \wedge (\sim S)$' is itself false. So we replace '?' by F:

U	V	W	S	$U \wedge V \wedge W$	$\sim S$	$U \wedge V \wedge W \wedge (\sim S)$	C	$[U \wedge V \wedge W \wedge (\sim S)] \to C$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
T	T	T	???	T	??	F	F	T

Now recall we have started with U, V, W (and hence $U \wedge V \wedge W$ also) being (assumed to be) true. So the only possibility is that '$\sim S$' is false. So we replace '??' by F:

U	V	W	S	$U \wedge V \wedge W$	$\sim S$	$U \wedge V \wedge W \wedge (\sim S)$	C	$[U \wedge V \wedge W \wedge (\sim S)] \to C$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
T	T	T	???	T	F	F	F	T

Hence S is true (under the assumption that U, V, W are true). So we replace '???' by T:

U	V	W	S	$U \wedge V \wedge W$	$\sim S$	$U \wedge V \wedge W \wedge (\sim S)$	C	$[U \wedge V \wedge W \wedge (\sim S)] \to C$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
T	T	T	T	T	F	F	F	T

This completes the argument for the statement 'if $U \wedge V \wedge W$ is true then S is true'.