MATH1050 Basics of logic in mathematics

1. **Statements**.

   A **(mathematical) statement** is a sentence, or a number of carefully worded inter-related sentences, (with mathematical content,) for which it is meaningful to say it is true or it is false.

   In the treatment below, all statements are placed on equal footing: there is prejudice neither against false statements such as '$1 + 1 = 3$', nor against true statements such as '$1 + 1 = 2$'.

   We accept **Aristotle's Law of the Excluded Middle**:

   - each statement is true or false, but not both.

   When a statement is known to be true, it is assigned the **truth value** T (for 'truth'); when it is known to be false, it is assigned the truth value F (for 'falsity').

2. **Decomposition of statements into blocks**.

   We can break up a statement into 'simpler blocks', each of them a statement in itself, when we delete from the original statement the words which indicate the 'logical relations' amongst the various blocks in the original statement, such as 'and', 'or', 'if', 'then', 'suppose', 'assume', 'let'.

   Examples.

   (a) Let ΔABC be a triangle. Suppose $\angle ACB$ is a right angle. Then $AB^2 = AC^2 + BC^2$.
   'Simpler blocks':
   (1) ΔABC is a triangle.
   (2) $\angle ACB$ is a right angle.
   (3) $AB^2 = AC^2 + BC^2$.

   (b) Let $x, y \in \mathbb{Z}$. Suppose x is divisible by y and y is divisible by x. Then $|x| = |y|$.
   'Simpler blocks':
   (1) $x \in \mathbb{Z}$.
   (2) $y \in \mathbb{Z}$.
   (3) x is divisible by y.
   (4) y is divisible by x.
   (5) $|x| = |y|$.

   (c) Let A, B be sets. $A \subset B \backslash (B \backslash A)$ iff $A \subset B$.
   'Simpler blocks':
   (1) A is a set.
   (2) B is a set.
   (3) $A \subset B \backslash (B \backslash A)$.
   (4) $A \subset B$.

   We may form a statement by joining one or several statements with words such as 'and', 'or', 'if', 'then', 'suppose', 'assume', 'let'. The original 'shorter' statements form 'blocks' which are 'logically related' amongst themselves in the new statement. For this reason, the words 'and', 'or', 'if', 'then', 'suppose', 'assume' et cetera are referred to as **'logical connectives'**. To emphasize that the new statement is a combination of the original 'shorter' statements, we call it a **compound statement**.

3. **Negation of a statement**.

   Given a statement P, we may form these two statements:

   - 'the statement P is true',
   - 'the statement P is not true'.

   We regard the statement P to be the same as 'the statement P is true': they are either true together, or false together.

   The statement 'the statement P is not true' is true exactly when P is false; it is false exactly when P is true.

   We denote the statement 'the statement P is not true' by $\sim P$ (pronounced as 'not P'), and call it the **negation** of P.

   Examples.

   (a) '$1 + 1 = 2$' is a statement.
   '$1 + 1 = 2$ is true', '$1 + 1 = 2$ is not true' are both statements.
   '$1 + 1 = 2$', '$1 + 1 = 2$ is true' are the same statement. Both are true.
   '$1 + 1 = 2$ is not true' is the negation of '$1 + 1 = 2$'. It is a false statement.

(b) '$1 = 2$' is a statement.

'$1 = 2$ is true', '$1 = 2$ is not true' are both statements.

'$1 = 2$', '$1 = 2$ is true' are the same statement. Both are false.

'$1 = 2$ is not true' is the negation of '$1 = 2$'. It is a true statement.

The relation between a statement P and its negation $\sim P$ is summarized in the table below:

P	$\sim P$
T	F
F	T

4. **Conjunctions and disjunctions of two statements**.

Given two statements P, Q, we may form the compound statements 'P and Q', 'P or Q'.

We denote the statement 'P and Q' by $P \wedge Q$ (pronounced as 'P wedge Q' or 'P and Q'), and call it the **conjunction** of P, Q.

We accept that $P \wedge Q$ is true exactly when both P, Q are true.

The relation amongst the statements P, Q, $P \wedge Q$ is summarized in the table below:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

We denote the statement 'P or Q' by $P \vee Q$ (pronounced as 'P vee Q' or 'P or Q'), and call it the **disjunction** of P, Q. We accept that $P \vee Q$ is true exactly when at least one of P, Q is true.

The relation amongst the statements P, Q, $P \vee Q$ is summarized in the table below:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Remark.

- The logical operation 'or' is different from the word 'or' in daily language. In daily life when we see 'coffee or tea' in the menu in a restaurant, we don't expect to be allowed three choices 'coffee and not tea', 'tea and not coffee', 'both coffee and tea'. In the context of logic, we are allowed to be 'greedy' when we see 'coffee or tea'.

5. **Truth table, and logical equivalence**.

A table such as the one relating $P, Q, P \wedge Q$ above is called a **truth table**.

We read such a table row-by-row. We usually put 'simpler blocks' on the left-hand-side (in this example P, Q), the 'simplest ones' on the extreme left, and compound statements (in this example $P \wedge Q$) formed from these blocks and 'logical connectives' (in this example $\wedge$) on the right-hand-side.

Given a compound statement formed by 'simpler' statements $P, Q, R, \cdots$ and 'logical connectives' $\sim, \wedge, \vee, \cdots$, we can compute the former's truth values for various truth values of $P, Q, R, \cdots$ 'step-by-step', working on one connective at a step, and present the computation in a truth table.

Examples.

(a) Truth table displaying the truth values of $\sim(P \vee Q)$.

P	Q	$P \vee Q$	$\sim(P \vee Q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

(b) Truth table displaying the truth values of $(\sim P) \wedge (\sim Q)$.

P	Q	$\sim P$	$\sim Q$	$(\sim P) \wedge (\sim Q)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(c) Truth table displaying the truth values of $\sim(P \vee Q)$, $(\sim P) \wedge (\sim Q)$ 'simultaneously'.

P	Q	$P \vee Q$	$\sim(P \vee Q)$	$\sim P$	$\sim Q$	$(\sim P) \wedge (\sim Q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

(d) Truth table displaying the truth values of $\sim(P \vee Q)$, $(\sim P) \vee Q$ 'simultaneously'.

P	Q	$P \vee Q$	$\sim(P \vee Q)$	$\sim P$	$(\sim P) \vee Q$
T	T	T	F	F	T
T	F	T	F	F	F
F	T	T	F	T	T
F	F	F	T	T	T

(e) Truth table displaying the truth values of $\sim(P \wedge Q)$, $(\sim P) \vee (\sim Q)$ 'simultaneously'.

P	Q	$P \wedge Q$	$\sim(P \wedge Q)$	$\sim P$	$\sim Q$	$(\sim P) \vee (\sim Q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Remarks.

(1) Refer to (c). It so happens that $\sim(P \vee Q)$, $(\sim P) \wedge (\sim Q)$ are both true or both false, (irrespective of the respective truth values of P, Q). For this reason, these two statements are the same as each other, and hence we say they are **logically equivalent** to each other.

(2) Refer to (d). $\sim(P \vee Q)$, $(\sim P) \vee Q$ are not logically equivalent. (Brackets are important!)

(3) Refer to (e). $\sim(P \wedge Q)$, $(\sim P) \vee (\sim Q)$ are logically equivalent to each other.

The logical equivalence of the respective pairs of statements in (c), (e) are known as **De Morgan's Laws** in logic.

Examples of applications.

- These statements are equivalent:
  - ∗ It is not true that both of a, b are rational.
  - ∗ It is not true that (a is rational and b is rational).
  - ∗ (a is irrational) or (b is irrational).
  - ∗ At least one of a, b is irrational.
- These statements are equivalent:
  - ∗ It is not true that at least one of a, b is rational.
  - ∗ It is not true that (a is rational or b is rational).
  - ∗ (a is irrational) and (b is irrational).
  - ∗ Each of a, b is irrational.

(4) Note the necessity of writing the brackets, to indicate how the statement is supposed to be read.

For instance, in (d), because $\sim(P \vee Q)$, $(\sim P) \vee Q$ are not logically equivalent, a reader will be confused by the chain of symbols '$\sim P \vee Q$',

Similarly, because $\sim(P \wedge Q)$, $(\sim P) \wedge Q$ are not logically equivalent, a reader will be confused by the chain of symbols '$\sim P \wedge Q$'

Examples (Continued).

(f) $(P \wedge Q) \vee R$, $(P \vee R) \wedge (Q \vee R)$ are logically equivalent. Justification:

P	Q	R	$P \wedge Q$	$(P \wedge Q) \vee R$	$P \vee R$	$Q \vee R$	$(P \vee R) \wedge (Q \vee R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	T	F	F
F	T	T	F	T	T	T	T
F	T	F	F	F	F	T	F
F	F	T	F	T	T	T	T
F	F	F	F	F	F	F	F

(g) $(P \vee Q) \wedge R$, $(P \wedge R) \vee (Q \wedge R)$ are logically equivalent. Justification:

P	Q	R	$P \vee Q$	$(P \vee Q) \wedge R$	$P \wedge R$	$Q \wedge R$	$(P \wedge R) \vee (Q \wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	F	T
T	F	F	T	F	F	F	F
F	T	T	T	T	F	T	T
F	T	F	T	F	F	F	F
F	F	T	F	F	F	F	F
F	F	F	F	F	F	F	F

(h) $(P \wedge Q) \vee R$, $P \wedge (Q \vee R)$ are not logically equivalent:

P	Q	R	$P \wedge Q$	$(P \wedge Q) \vee R$	$Q \vee R$	$P \wedge (Q \vee R)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	F	T	T	T
T	F	F	F	F	F	F
F	T	T	F	T	T	F
F	T	F	F	F	T	F
F	F	T	F	T	T	F
F	F	F	F	F	F	F

Remarks.

(5) Refer to (f), (g). The logical equivalence given are known as the **Distributive Laws for conjunction and disjunction** in logic.

(6) Refer to (h). Note that $(P \wedge Q) \vee R$, $P \wedge (Q \vee R)$ are not logically equivalent.
Hence it is important to put appropriate brackets into a statement to indicate how the statement is supposed to be read.
Example. '$(x > 2$ and $x < 4)$ or $x > 1$', '$x > 2$ and $(x < 4$ or $x > 1)$' mean different things. The first is the same as '$x > 1$', while the second is the same as '$x > 2$'.

Examples of pairs of logically equivalent statements.

- **De Morgan's Laws**.
  $\sim(P \vee Q)$, $(\sim P) \wedge (\sim Q)$ are logically equivalent.
  $\sim(P \wedge Q)$, $(\sim P) \vee (\sim Q)$ are logically equivalent.
- **Distributive Laws for conjunction and disjunction**.
  $(P \vee Q) \wedge R$, $(P \wedge R) \vee (Q \wedge R)$ are logically equivalent.
  $(P \wedge Q) \vee R$, $(P \vee R) \wedge (Q \vee R)$ are logically equivalent.
- **Law of Double Negative Elimination**:
  P, $\sim(\sim P)$ are logically equivalent.
- **Law of Commutativity of Conjunction**:
  $P \wedge Q$, $Q \wedge P$ are logically equivalent.
- **Law of Commutativity of Disjunction**:
  $P \vee Q$, $Q \vee P$ are logically equivalent.

- **Law of Associativity of Conjunction**:

  $(P \wedge Q) \wedge R$, $P \wedge (Q \wedge R)$ are logically equivalent.

- **Law of Associativity of Disjunction**:

  $(P \vee Q) \vee R$, $P \vee (Q \vee R)$ are logically equivalent.

6. **Conditionals**.

   Given two statements P, Q, we may form the compound statement 'if P then Q', using the pair of words 'if', 'then' simultaneously. We denote the statement 'if P then Q' by $P \to Q$ (pronounced as 'P arrow Q' or 'if P then Q'), and call it the **conditional** from P to Q. We call P the **assumption** in $P \to Q$, and Q the **conclusion** in $P \to Q$. The statement $P \to Q$ is true except when P is true and Q is false.

   The relation amongst $P, Q, P \to Q$ is summarized in the truth table below:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

   These are various 'wordy' formulations for $P \to Q$:

   (a) 'P only if Q'.

   (b) 'Suppose P. Then Q.'

   (c) 'Q is necessary for P'. 'Q is a **necessary condition** for P'.

   (d) 'P is sufficient for Q'. 'P is a **sufficient condition** for Q'.

   (e) 'Assuming/Given/Provided that P is true, Q is true'.

7. **Another way to see what conditionals are**.

   Consider the statements $P \to Q$, $(\sim P) \vee Q$, and the latter's 'double negation' $\sim[P \wedge (\sim Q)]$.

   We draw a truth table showing truth values of $P \to Q$, $(\sim P) \vee Q$, $\sim[P \wedge (\sim Q)]$ simultaneously:

P	Q	$P \to Q$	$\sim P$	$\sim Q$	$(\sim P) \vee Q$	$P \wedge (\sim Q)$	$\sim[P \wedge (\sim Q)]$
T	T	T	F	F	T	F	T
T	F	F	F	T	F	T	F
F	T	T	T	F	T	F	T
F	F	T	T	T	T	F	T

   Therefore $P \to Q$, $(\sim P) \vee Q$, $\sim[P \wedge (\sim Q)]$ are logically equivalent. In all practical purposes, the statement $P \to Q$ and the negation of the statement $P \wedge (\sim Q)$ are regarded to be the same statement. This turns out to be the logical foundation of the proof-by-contradiction method.

   Remark. Some people choose to define $P \to Q$ as $(\sim P) \vee Q$.

8. **Converse, contrapostive and inverse of a conditional**.

   Consider the statements $P, Q, P \to Q$.

   We may form the statements $Q \to P$, $(\sim Q) \to (\sim P)$, $(\sim P) \to (\sim Q)$.

   - The statement $Q \to P$ is called the **converse** of the conditional $P \to Q$.
   - The statement $(\sim Q) \to (\sim P)$ is called the **contrapositive** of the conditional $P \to Q$.
   - The statement $(\sim P) \to (\sim Q)$ is called the **inverse** of the conditional $P \to Q$.

   We draw a truth table showing the truth values of $P \to Q$, $Q \to P$, $(\sim Q) \to (\sim P)$, $(\sim P) \to (\sim Q)$ simultaneously:

P	Q	$P \to Q$	$Q \to P$	$\sim P$	$\sim Q$	$(\sim P) \to (\sim Q)$	$(\sim Q) \to (\sim P)$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

   Therefore,

- $P \to Q$, $(\sim Q) \to (\sim P)$ are logically equivalent,
- $P \to Q$, $Q \to P$ are not logically equivalent,
- $P \to Q$, $(\sim P) \to (\sim Q)$ are not logically equivalently, and
- $Q \to P$, $(\sim P) \to (\sim Q)$ are logically equivalent.

Two key remarks.

- A statement and its contrapositive are logically equivalent, and in all pratical purposes may be regarded to be the same statement. So to justify one of them is the same as to justify the other. This is the logical foundation of the 'contrapositive proof'.
- A statement and its converse are not logically equivalent. It can happen that both are true. It can happen that both are false. It can also happen that one is true while the other is false.
  Examples.
  (a) Let P stand for 'the quadrilateral $ABCD$ is a square', and Q stand for 'all angles of the quadrilateral $ABCD$ are right angles'.
  $P \to Q$ is true.
  $Q \to P$ is false.
  (b) Let P stand for '$\triangle ABC$ is equilateral', and Q stand for 'all three angles in $\triangle ABC$ are equal to each other'.
  $P \to Q$ is true.
  $Q \to P$ is true.
  (c) Let P stand for '$\triangle ABC$ is an isosceles triangle', and Q stand for '$\triangle ABC$ is a right-angle triangle'.
  $P \to Q$ is false.
  $Q \to P$ is false.

In Euclidean geometry, there are a lot of pairs of conditionals and converses which are both true. Examples:

(a) Parallel Postulate and its converse (Fifth Postulate, and Proposition 27 of Book I, *Euclid's Elements*).

(b) Pythagoras' Theorem and its converse (Propositions 47, 48 of Book I, *Euclid's Elements*).

(c) Thales' Theorem (Proposition 31 of Book III, *Euclid's Elements*) and its converse.

In your *analysis* course, you will find a lot of conditionals which are true but whose respective converses are false. These are the simplest examples:

(a) (...) Suppose f is differentiable at c. Then f is continuous at c.

(b) (...) Suppose f is continuous on $[a, b]$. Then f is integrable on $[a, b]$.

9. **Biconditionals**.
Given two statements P, Q, we may form the compound statement 'P if and only if Q', using the phrase 'if and only if', or in short-hand 'P iff Q'. We denote the statement 'P iff Q' by $P \leftrightarrow Q$ (pronounced as 'P double-arrow Q' or 'P if and only if Q'), and call it the **biconditional** from P to Q. The statement $P \leftrightarrow Q$ is true exactly when P, Q are both true or both false.
The relation amongst $P, Q, P \leftrightarrow Q$ is summarized in the truth table below:

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Various 'wordy' formulations for $P \leftrightarrow Q$:

(a) 'P is necessary and sufficient for Q'.

(b) 'P is a **necessary and sufficient condition** for Q'.

(c) 'Q is a necessary and sufficient condition for P'.

(d) 'P, Q are **(logically) equivalent** to each other'.

$P \leftrightarrow Q$, $(P \to Q) \wedge (Q \to P)$ are logically equivalent:

P	Q	$P \leftrightarrow Q$	$P \to Q$	$Q \to P$	$(P \to Q) \wedge (Q \to P)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

To justify '$P \leftrightarrow Q$ is true' is the same as to justify '$(P \to Q) \wedge (Q \to P)$ is true', which is the same as to justify both of '$P \to Q$ is true','$Q \to P$ is true'.

10. **Tautologies, contradictions and contingent statements**.

Consider a compound statement $\Sigma(P, Q, R, \cdots)$ formed by 'connecting' a number of statements $P, Q, R, \cdots$ with the 'logical connectives' $\sim, \vee, \wedge, \to, \leftrightarrow$.

- The statement $\Sigma(P, Q, R, \cdots)$ is called a **tautology** exactly when it is always true irrespective of the truth values of $P, Q, R, \cdots$.
- The statement $\Sigma(P, Q, R, \cdots)$ is called a **contradiction** exactly when it is always false irrespective of the truth values of $P, Q, R, \cdots$.
- The statement $\Sigma(P, Q, R, \cdots)$ is called a **contingent statement** exactly when it is neither a tautology nor a contradiction.

Examples of tautologies.

(a) $P \vee (\sim P)$.

(b) $P \to P$.

(c) $P \leftrightarrow [\sim(\sim P)]$.

(d) $(P \wedge Q) \to P$.

(e) $P \to (P \vee Q)$.

(f) $[P \wedge (Q \vee R)] \leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$.

Examples of contradictions.

(a) $P \wedge (\sim P)$.

(b) $P \leftrightarrow (\sim P)$.

Examples of contingent statements.

(a) P.

(b) $P \wedge Q$.

(c) $P \vee Q$.

(d) $P \to Q$.

(e) $(\sim P) \to P$.

(f) $P \to (P \wedge Q)$.

(g) $(P \vee Q) \to P$.

11. **Another view on logical equivalence**.

Given statements $P, Q, \cdots$, we form two compound statements $\Sigma(P, Q, \cdots), \Sigma'(P, Q, \cdots)$, and then further form the compound statement $\Sigma(P, Q, \cdots) \leftrightarrow \Sigma'(P, Q, \cdots)$. In general such a statement needs not be a tautology. It is a tautology exactly when $\Sigma(P, Q, \cdots), \Sigma'(P, Q, \cdots)$ are logically equivalent.

Examples of logical equivalence.

(a) Law of Double Negation: $P \leftrightarrow [\sim(\sim P)]$.

(b) Distributive Law: $[(P \wedge Q) \vee R] \leftrightarrow [(P \vee R) \wedge (Q \vee R)]$, $[(P \vee Q) \wedge R] \leftrightarrow [(P \wedge R) \vee (Q \wedge R)]$.

(c) De Morgan's Law: $[\sim(P \wedge Q)] \leftrightarrow [(\sim P) \vee (\sim Q)]$, $[\sim(P \vee Q)] \leftrightarrow [(\sim P) \wedge (\sim Q)]$.

(d) $(P \to Q) \leftrightarrow [(\sim P) \vee Q]$.

(e) $(P \to Q) \leftrightarrow \{\sim[P \wedge (\sim Q)]\}$.

(f) $(P \to Q) \leftrightarrow [(\sim Q) \to (\sim P)]$.

12. **Rules of inference**.

Given statements $P, Q, \cdots$, we form compound statements $\Sigma_1(P, \cdots)$, ..., $\Sigma_n(P, \cdots)$, $\Sigma'(P, \cdots)$, and then further form the compound statement $(\Sigma_1(P, \cdots) \wedge \cdots \wedge \Sigma_n(P, \cdots)) \to \Sigma'(P, \cdots)$. In general such a statement needs not be a tautology. It is called a **rule of inference** exactly when it is a tautology. It is usually presented in the table form

$$\begin{array}{c} \Sigma_1(P, Q, \cdots) \\ \Sigma_2(P, Q, \cdots) \\ \vdots \\ \Sigma_n(P, Q, \cdots) \\ \hline \Sigma'(P, Q, \cdots) \end{array}$$

Examples of rules of inference.

(a) **Modus Ponens**: $[(P \to Q) \wedge P] \to Q$.

(b) **Modus Tollendo Ponens**: $[(P \vee Q) \wedge (\sim P)] \to Q$, $[(P \vee Q) \wedge (\sim Q)] \to P$.

(c) **Modus Tollens**: $[(P \to Q) \wedge (\sim Q)] \to (\sim P)$.

(d) **Hypothetical syllogism**: $[(P \to Q) \wedge (Q \to R)] \to (P \to R)$.

(e) **Biconditional-conditional**: $(P \leftrightarrow Q) \to (P \to Q)$, $(P \leftrightarrow Q) \to (Q \to P)$.

(f) **Conditional-biconditional**: $[(P \to Q) \wedge (Q \to P)] \to (P \leftrightarrow Q)$.

(g) **Simplification**: $(P \wedge Q) \to P$, $(P \wedge Q) \to Q$.

(h) **Addition**: $P \to (P \vee Q)$, $Q \to (P \vee Q)$.

(i) **Repetition**: $P \to P$.

(j) **Double negation**: $[\sim(\sim P)] \to P$.

(k) **Adjunction**: $(P \wedge Q) \to (P \wedge Q)$.

(l) **Constructive dilemma**: $[(P \to Q) \wedge (R \to S) \wedge (P \vee R)] \to (Q \vee S)$.

(m) **Idempotency of entailment**: $[P \to (P \to Q)] \to (P \to Q)$.

(n) **Monotonicity of entailment**: $(P \to Q) \to [(P \wedge R) \to Q]$.